Economic Model Predictive Control

Mario Zanon, Alberto Bemporad

Outline

MPC

Optimal Control Problem

$$V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N}$$

$$\text{s.t.}$$

MPC

Optimal Control Problem

$$V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N}$$

$$\text{s.t.} \quad x_0 = \hat{x}_0, \qquad \text{Initial condition}$$

MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} & \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics}
\end{aligned}
$$

MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} & \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics} \\
& h(x_k, u_k) \geq 0, && \text{Path constraints}
\end{aligned}
$$

MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} & \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics} \\
& h(x_k, u_k) \geq 0, && \text{Path constraints} \\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$

MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) + V_f(x_N) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics} \\
& h(x_k, u_k) \geq 0, && \text{Path constraints} \\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$

MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) + V_f(x_N) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics} \\
& h(x_k, u_k) \geq 0, && \text{Path constraints} \\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$

At each sampling time:

- get the initial state $\hat{x}_0$

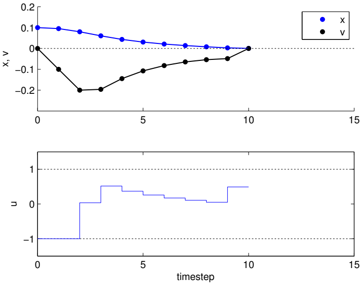
MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0,u_0,\ldots,x_N} \quad & \sum_{k=0}^{N-1} l(x_k,u_k) + V_f(x_N) && \text{(Quadratic) stage cost}\\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition}\\
& x_{k+1} = f(x_k,u_k), && \text{System dynamics}\\
& h(x_k,u_k) \geq 0, && \text{Path constraints}\\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$

At each sampling time:

- get the initial state $\hat{x}_0$
- solve the MPC OCP



MPC

Optimal Control Problem

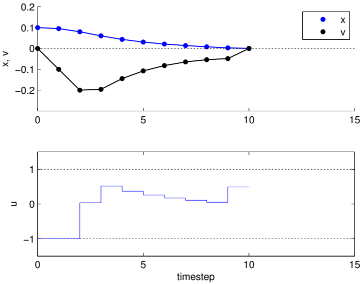
$$V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} \sum_{k=0}^{N-1} l(x_k, u_k) + V_f(x_N) \qquad \text{(Quadratic) stage cost}$$

$$\text{s.t.} \quad x_0 = \hat{x}_0, \qquad \text{Initial condition}$$

$$x_{k+1} = f(x_k, u_k), \qquad \text{System dynamics}$$

$$h(x_k, u_k) \geq 0, \qquad \text{Path constraints}$$

$$x_N = \mathbb{X}_f. \qquad \text{Terminal constraint}$$



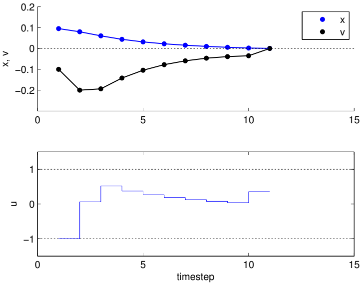
At each sampling time:

- get the initial state $\hat{x}_0$
- solve the MPC OCP
- apply the first control u_0^*

MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) + V_f(x_N) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics} \\
& h(x_k, u_k) \geq 0, && \text{Path constraints} \\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$



At each sampling time:

- get the initial state $\hat{x}_0$
- solve the MPC OCP
- apply the first control u_0^*

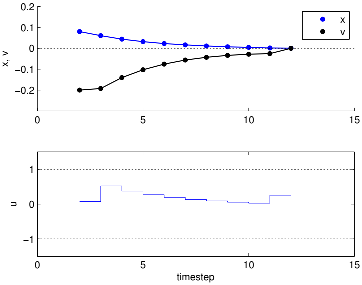
MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) + V_f(x_N) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics} \\
& h(x_k, u_k) \geq 0, && \text{Path constraints} \\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$

At each sampling time:

- get the initial state $\hat{x}_0$
- solve the MPC OCP
- apply the first control u_0^*



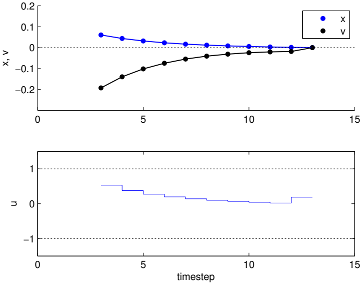
MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) + V_f(x_N) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics} \\
& h(x_k, u_k) \geq 0, && \text{Path constraints} \\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$

At each sampling time:

- get the initial state $\hat{x}_0$
- solve the MPC OCP
- apply the first control u_0^*



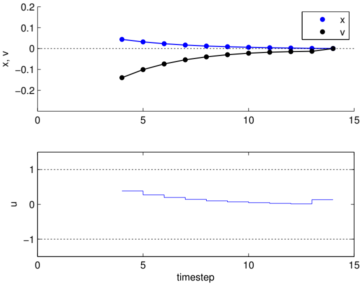
MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0,u_0,\dots,x_N} \quad & \sum_{k=0}^{N-1} l(x_k,u_k) + V_f(x_N) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k,u_k), && \text{System dynamics} \\
& h(x_k,u_k) \geq 0, && \text{Path constraints} \\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$

At each sampling time:

- get the initial state $\hat{x}_0$
- solve the MPC OCP
- apply the first control u_0^*



MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0,u_0,\ldots,x_N} \quad & \sum_{k=0}^{N-1} l(x_k,u_k) + V_f(x_N) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k,u_k), && \text{System dynamics} \\
& h(x_k,u_k) \geq 0, && \text{Path constraints} \\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$

If $V_N(x)$ Lyapunov Function
$\Rightarrow$ asymptotic stability

At each sampling time:

- get the initial state $\hat{x}_0$
- solve the MPC OCP
- apply the first control u_0^*

MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) + V_f(x_N) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics} \\
& h(x_k, u_k) \geq 0, && \text{Path constraints} \\
& x_N = \mathbb{X}_f. && \text{Terminal constraint}
\end{aligned}
$$

If $V_N(x)$ Lyapunov Function
$\Rightarrow$ asymptotic stability

$$\alpha_1(\|x - x_{\mathrm{s}}\|) \leq V_N(x) \leq \alpha_2(\|x - x_{\mathrm{s}}\|)$$
$$V_N(f(x, u)) - V_N(x) \leq -\alpha_3(\|x - x_{\mathrm{s}}\|)$$

At each sampling time:

- get the initial state $\hat{x}_0$
- solve the MPC OCP
- apply the first control u_0^*

MPC

Optimal Control Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 = \hat{x}_0, && \text{Initial condition} \\
& x_{k+1} = f(x_k, u_k), && \text{System dynamics} \\
& h(x_k, u_k) \geq 0, && \text{Path constraints} \\
& x_N = x_\mathrm{s}. && \text{Terminal constraint}
\end{aligned}
$$

If $V_N(x)$ Lyapunov Function
$\Rightarrow$ asymptotic stability

$$\alpha_1(\|x - x_\mathrm{s}\|) \leq V_N(x) \leq \alpha_2(\|x - x_\mathrm{s}\|)$$
$$V_N(f(x,u)) - V_N(x) \leq -\alpha_3(\|x - x_\mathrm{s}\|)$$

At each sampling time:

- get the initial state $\hat{x}_0$
- solve the MPC OCP
- apply the first control u_0^*

Lyapunov stability

Tracking MPC: $l(x_s, u_s) = 0$ and $\exists\ \alpha \in \mathcal{K}$ s.t. $\alpha(\|x - x_s\|) \leq l(x, u),\ \forall u \in \mathbb{U}$

Lyapunov stability

Tracking MPC: $l(x_\mathrm{s}, u_\mathrm{s}) = 0$ and $\exists\, \alpha \in \mathcal{K}$ s.t. $\alpha(\|x - x_\mathrm{s}\|) \leq l(x, u),\ \forall u \in \mathbb{U}$

$$V_N(\hat{x}_0) = \underbrace{l(x_0, u_0)}_{\geq \alpha(\|x - x_\mathrm{s}\|)} + \underbrace{\sum_{k=0}^{N-1} l(x_k, u_k)}_{\geq 0} \geq \alpha(\|x - x_\mathrm{s}\|)$$

Lyapunov stability

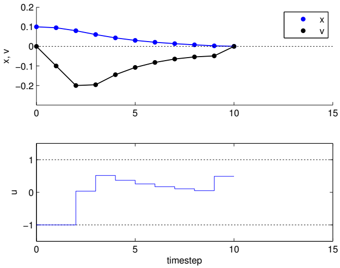
Tracking MPC: $l(x_s, u_s) = 0$ and $\exists\, \alpha \in \mathcal{K}$ s.t. $\alpha(\|x - x_s\|) \leq l(x, u),\ \forall u \in \mathbb{U}$

$$V_N(\hat{x}_0) = \underbrace{l(x_0, u_0)}_{\geq \alpha(\|x - x_s\|)} + \underbrace{\sum_{k=0}^{N-1} l(x_k, u_k)}_{\geq 0} \geq \alpha(\|x - x_s\|)$$

$V_N(\hat{x}_0) \leq \alpha_2(\|x - x_s\|)$ Controllability assumption

Lyapunov stability

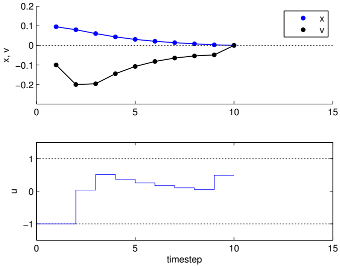
Tracking MPC: $l(x_\mathrm{s}, u_\mathrm{s}) = 0$ and $\exists\, \alpha \in \mathcal{K}$ s.t. $\alpha(\|x - x_\mathrm{s}\|) \leq l(x, u),\ \forall u \in \mathbb{U}$



$V_N(\hat{x}_0)$

Lyapunov stability

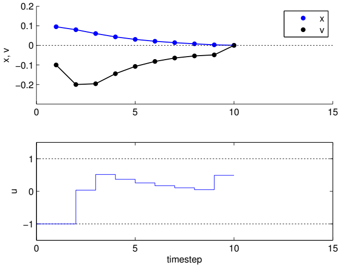
Tracking MPC: $l(x_s, u_s) = 0$ and $\exists\, \alpha \in \mathcal{K}$ s.t. $\alpha(\|x - x_s\|) \leq l(x, u),\ \forall u \in \mathbb{U}$



$$V_{N-1}(f(\hat{x}_0, u_0^*))$$

Lyapunov stability

Tracking MPC: $l(x_s, u_s) = 0$ and $\exists\, \alpha \in \mathcal{K}$ s.t. $\alpha(\|x - x_s\|) \leq l(x, u),\ \forall u \in \mathbb{U}$

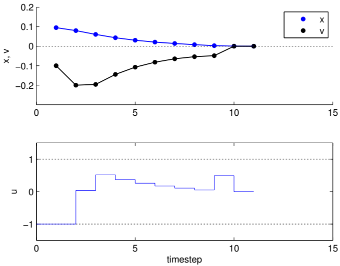


$$V_{N-1}(f(\hat{x}_0, u_0^*))$$

$$V_{N-1}(f(\hat{x}_0, u_0^*)) = V_N(\hat{x}_0) - l(\hat{x}_0, u_0^*)$$

Lyapunov stability

Tracking MPC: $l(x_s, u_s) = 0$ and $\exists\, \alpha \in \mathcal{K}$ s.t. $\alpha(\|x - x_s\|) \leq l(x, u),\ \forall u \in \mathbb{U}$



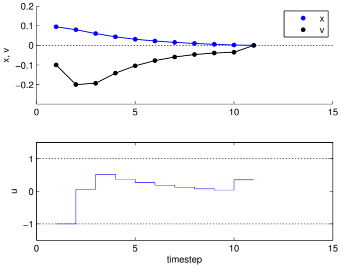
$$V_{N-1}(f(\hat{x}_0, u_0^*)) + l(x_s, u_s)$$

$$V_{N-1}(f(\hat{x}_0, u_0^*)) = V_N(\hat{x}_0) - l(\hat{x}_0, u_0^*)$$

$$V_N(\hat{x}_0) - l(\hat{x}_0, u_0^*) + \underbrace{l(x_s, u_s)}_{=0}$$

Lyapunov stability

Tracking MPC: $l(x_s, u_s) = 0$ and $\exists\, \alpha \in \mathcal{K}$ s.t. $\alpha(\|x - x_s\|) \leq l(x, u),\ \forall u \in \mathbb{U}$



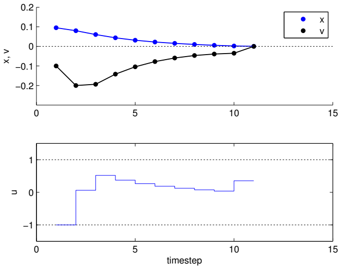
$$V_N(f(\hat{x}_0, u_0^*))$$

$$\begin{aligned} V_{N-1}(f(\hat{x}_0, u_0^*)) &= V_N(\hat{x}_0) - l(\hat{x}_0, u_0^*) \\ V_N(f(\hat{x}_0, u_0^*)) &\leq V_N(\hat{x}_0) - l(\hat{x}_0, u_0^*) + \underbrace{l(x_s, u_s)}_{=0} \end{aligned}$$

Different terminal conditions can still enforce stability

Lyapunov stability

Tracking MPC: $l(x_s, u_s) = 0$ and $\exists\, \alpha \in \mathcal{K}$ s.t. $\alpha(\|x - x_s\|) \leq l(x, u),\ \forall u \in \mathbb{U}$



$$V_N(f(\hat{x}_0, u_0^*))$$

$$V_{N-1}(f(\hat{x}_0, u_0^*)) = V_N(\hat{x}_0) - l(\hat{x}_0, u_0^*)$$

$$V_N(f(\hat{x}_0, u_0^*)) \leq V_N(\hat{x}_0) - l(\hat{x}_0, u_0^*) + \underbrace{l(x_s, u_s)}_{=0}$$

$$V_N(f(\hat{x}_0, u_0^*)) - V_N(\hat{x}_0) \leq -l(\hat{x}_0, u_0^*) \leq -\alpha(\|x - x_s\|)$$

Do we always want to track?

Do we always want to track?

Do we always want to track?

Do we always want to track?

Economic MPC

Do we always want to track?

No!

Economic MPC

Do we always want to track?

No!

Then why do we track?

Do we always want to track?

No!

Then why do we track?

- It works

Do we always want to track?

No!

Then why do we track?

- It works
- We have been doing it since a long time

Do we always want to track?

No!

Then why do we track?

- It works
- We have been doing it since a long time
- We have stability guarantees

Economic MPC

Do we always want to track?

No!

Then why do we track?

- It works
- We have been doing it since a long time
- We have stability guarantees

What about Economic MPC?

Do we always want to track?

No!

Then why do we track?

- It works
- We have been doing it since a long time
- We have stability guarantees

What about Economic MPC?

- Increased "economic" gain

Do we always want to track?

No!

Then why do we track?

- It works
- We have been doing it since a long time
- We have stability guarantees

What about Economic MPC?

- Increased "economic" gain
- Difficult to prove stability (2008-)

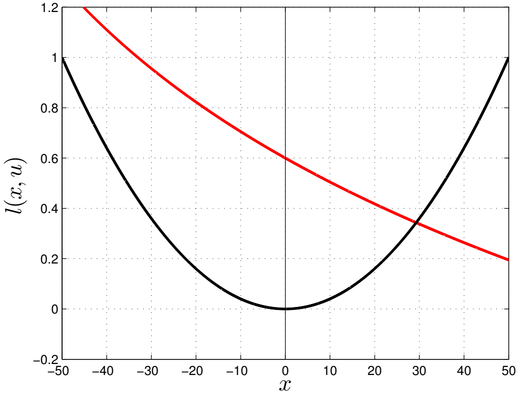
Economic vs Tracking

Economic vs Tracking

Stage cost: **Tracking** vs **Economic**

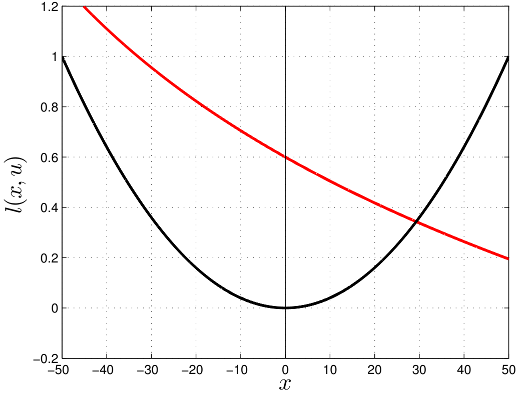
Economic vs Tracking

Stage cost: **Tracking** vs **Economic**



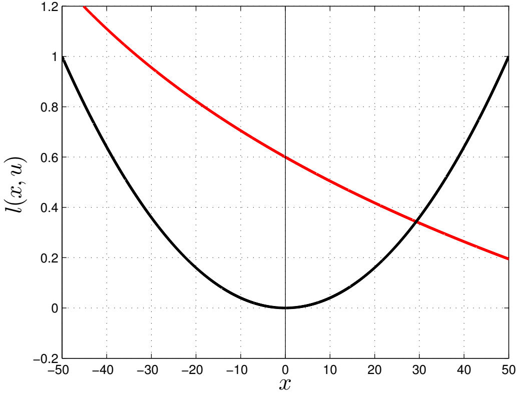
Economic vs Tracking

Stage cost: **Tracking** vs **Economic** ($:\neq$ Tracking, $\nexists\ \alpha(\|x - x_{\mathrm{s}}\|) \leq l(x, u)$)



Economic vs Tracking

Stage cost: **Tracking** vs **Economic** (:$\neq$ Tracking, $\nexists\ \alpha(\|x - x_{\mathrm{s}}\|) \leq l(x, u)$)



The classical stability theory does not apply!

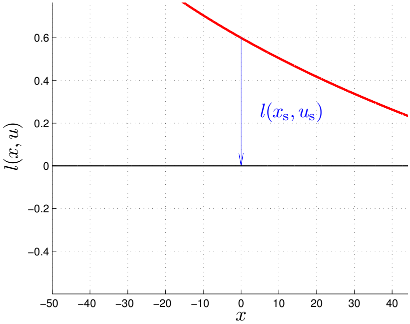
Economic Stage Cost

$$\text{Steady state:} \quad (x_\mathrm{s}, u_\mathrm{s}) = \min_{x,u} \; l(x,u) \qquad \text{s.t.} \;\; x = f(x,u)$$

Economic Stage Cost

$$\text{Steady state:} \quad (x_\text{s}, u_\text{s}) = \min_{x,u} \; l(x,u) \qquad \text{s.t.} \;\; x = f(x,u)$$

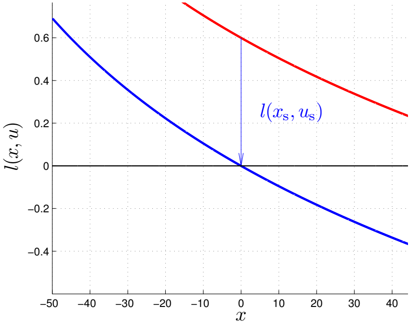
$l(x,u)$ $\quad$ $l(x_\text{s}, u_\text{s})$



Economic Stage Cost

Steady state: $(x_s, u_s) = \min_{x,u} \; l(x,u) \qquad \text{s.t.} \;\; x = f(x,u)$

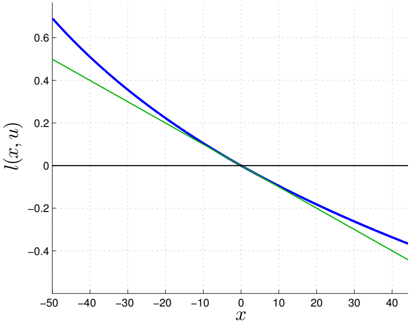
$l(x,u) - l(x_s, u_s)$



Economic Stage Cost

Steady state: $$(x_s, u_s) = \min_{x,u} \quad l(x,u) \qquad \text{s.t.} \quad x = f(x,u)$$

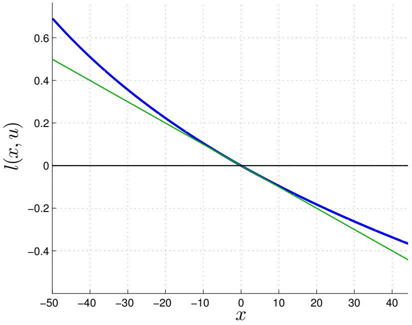
$$l(x,u) - l(x_s, u_s)$$



Economic Stage Cost

Steady state: $(x_\mathrm{s}, u_\mathrm{s}) = \min_{x,u} \; l(x,u) \quad \text{s.t.} \; x = f(x,u)$

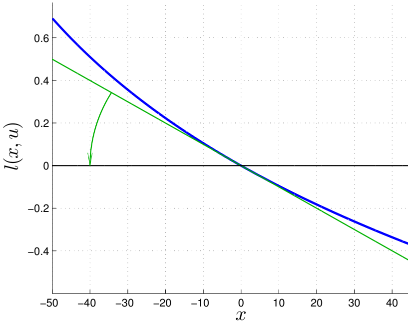
$$l(x,u) - l(x_\mathrm{s}, u_\mathrm{s}) \qquad \underbrace{\lambda_\mathrm{s}^\top}_{\text{Lagrange multiplier}} \qquad (x - f(x,u))$$



Economic Stage Cost

Steady state: $$(x_\mathrm{s}, u_\mathrm{s}) = \min_{x,u} \; l(x,u) \qquad \text{s.t.} \;\; x = f(x,u)$$

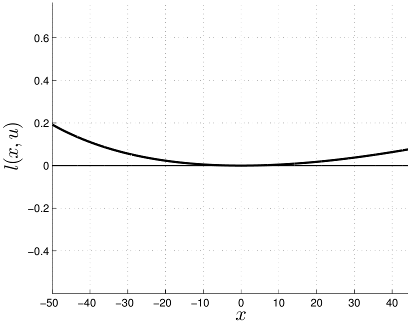
$$l(x,u) - l(x_\mathrm{s}, u_\mathrm{s}) + \underbrace{\lambda_\mathrm{s}^\top}_{\text{Lagrange multiplier}} (x - f(x,u))$$



Economic Stage Cost

Steady state: $(x_s, u_s) = \min_{x,u} \; l(x,u) \quad \text{s.t.} \quad x = f(x,u)$

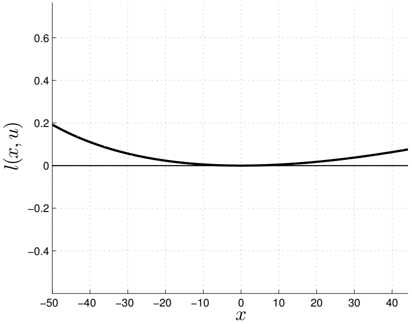
$$L(x,u) = l(x,u) - l(x_s,u_s) + \underbrace{\lambda_s^\top}_{\text{Lagrange multiplier}} (x - f(x,u))$$



Economic Stage Cost

Steady state: $(x_s, u_s) = \min_{x,u} \; l(x,u) \quad \text{s.t.} \; x = f(x,u)$

Rotated cost [Diehl et al. 2011]: $L(x,u) = l(x,u) - l(x_s,u_s) + \underbrace{\lambda_s^\top}_{\text{Lagrange multiplier}} (x - f(x,u))$

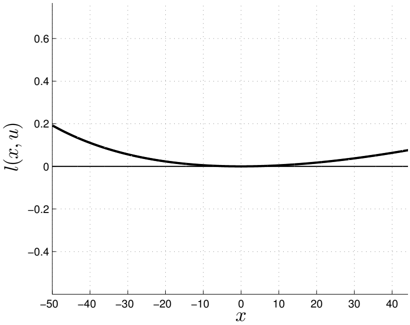


Economic Stage Cost

Steady state: $(x_s, u_s) = \min_{x,u} \; l(x,u) \quad \text{s.t.} \;\; x = f(x,u)$

Rotated cost: $L(x,u) = l(x,u) - l(x_s,u_s) + \lambda(x) - \lambda(f(x,u))$

[Amrit et al. 2011]



Rotated MPC Problem

Original Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k, u_k), \\
& h(x_k, u_k) \geq 0, \\
& x_N = x_\mathrm{s}.
\end{aligned}
$$

Rotated Problem

$$
\begin{aligned}
\bar{V}_N(\hat{x}_0) := \min_{x_0, u_0, \ldots, x_N} \quad & \sum_{k=0}^{N-1} L(x_k, u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k, u_k), \\
& h(x_k, u_k) \geq 0, \\
& x_N = x_\mathrm{s}.
\end{aligned}
$$

Rotated MPC Problem

Original Problem

$$\begin{aligned} V_N(\hat{x}_0) := \min_{x_0,u_0,\dots,x_N} \quad & \sum_{k=0}^{N-1} l(x_k,u_k) \\ \text{s.t.} \quad & x_0 = \hat{x}_0, \\ & x_{k+1} = f(x_k,u_k), \\ & h(x_k,u_k) \geq 0, \\ & x_N = x_{\mathrm{s}}. \end{aligned}$$

Rotated Problem

$$\begin{aligned} \bar{V}_N(\hat{x}_0) := \min_{x_0,u_0,\dots,x_N} \quad & \sum_{k=0}^{N-1} L(x_k,u_k) \\ \text{s.t.} \quad & x_0 = \hat{x}_0, \\ & x_{k+1} = f(x_k,u_k), \\ & h(x_k,u_k) \geq 0, \\ & x_N = x_{\mathrm{s}}. \end{aligned}$$

$$\sum_{k=0}^{N-1} L(x,u) = \sum_{k=0}^{N-1} l(x_k,u_k) + \sum_{k=0}^{N-1} \lambda(x_k) - \sum_{k=0}^{N-1} \lambda(f(x_k,u_k)) - \sum_{k=0}^{N-1} l(x_{\mathrm{s}},u_{\mathrm{s}})$$

Rotated MPC Problem

Original Problem

$$V_N(\hat{x}_0) := \min_{x_0, u_0, \dots, x_N} \sum_{k=0}^{N-1} l(x_k, u_k)$$

$$\text{s.t.} \quad x_0 = \hat{x}_0, \quad x_{k+1} = f(x_k, u_k), \quad h(x_k, u_k) \geq 0, \quad x_N = x_{\mathrm{s}}.$$

Rotated Problem

$$\bar{V}_N(\hat{x}_0) := \min_{x_0, u_0, \dots, x_N} \sum_{k=0}^{N-1} L(x_k, u_k)$$

$$\text{s.t.} \quad x_0 = \hat{x}_0, \quad x_{k+1} = f(x_k, u_k), \quad h(x_k, u_k) \geq 0, \quad x_N = x_{\mathrm{s}}.$$

$$\begin{aligned}\sum_{k=0}^{N-1} L(x, u) &= \sum_{k=0}^{N-1} l(x_k, u_k) + \sum_{k=0}^{N-1} \lambda(x_k) - \sum_{k=0}^{N-1} \lambda(f(x_k, u_k)) - \sum_{k=0}^{N-1} l(x_{\mathrm{s}}, u_{\mathrm{s}}) \\ &= \sum_{k=0}^{N-1} l(x_k, u_k) + \sum_{k=0}^{N-1} \lambda(x_k) - \sum_{k=0}^{N-1} \lambda(x_{k+1}) - N\, l(x_{\mathrm{s}}, u_{\mathrm{s}})\end{aligned}$$

Rotated MPC Problem

Original Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0,u_0,\dots,x_N} \quad & \sum_{k=0}^{N-1} l(x_k,u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k,u_k), \\
& h(x_k,u_k) \geq 0, \\
& x_N = x_\mathrm{s}.
\end{aligned}
$$

Rotated Problem

$$
\begin{aligned}
\bar{V}_N(\hat{x}_0) := \min_{x_0,u_0,\dots,x_N} \quad & \sum_{k=0}^{N-1} L(x_k,u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k,u_k), \\
& h(x_k,u_k) \geq 0, \\
& x_N = x_\mathrm{s}.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{N-1} L(x,u) &= \sum_{k=0}^{N-1} l(x_k,u_k) + \sum_{k=0}^{N-1} \lambda(x_k) - \sum_{k=0}^{N-1} \lambda(f(x_k,u_k)) - \sum_{k=0}^{N-1} l(x_\mathrm{s},u_\mathrm{s}) \\
&= \sum_{k=0}^{N-1} l(x_k,u_k) + \sum_{k=0}^{N-1} \lambda(x_k) - \sum_{k=0}^{N-1} \lambda(x_{k+1}) - N\, l(x_\mathrm{s},u_\mathrm{s}) \\
&= \sum_{k=0}^{N-1} l(x_k,u_k) + \underbrace{\lambda(x_0) - \lambda(x_N) - N\, l(x_\mathrm{s},u_\mathrm{s})}_{\text{constant}}
\end{aligned}
$$

Rotated MPC Problem

Original Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0,u_0,\ldots,x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k, u_k), \\
& h(x_k, u_k) \geq 0, \\
& x_N = x_\mathrm{s}.
\end{aligned}
$$

Rotated Problem

$$
\begin{aligned}
\bar{V}_N(\hat{x}_0) := \min_{x_0,u_0,\ldots,x_N} \quad & \sum_{k=0}^{N-1} L(x_k, u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k, u_k), \\
& h(x_k, u_k) \geq 0, \\
& x_N = x_\mathrm{s}.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{N-1} L(x, u) &= \sum_{k=0}^{N-1} l(x_k, u_k) + \sum_{k=0}^{N-1} \lambda(x_k) - \sum_{k=0}^{N-1} \lambda(f(x_k, u_k)) - \sum_{k=0}^{N-1} l(x_\mathrm{s}, u_\mathrm{s}) \\
&= \sum_{k=0}^{N-1} l(x_k, u_k) + \sum_{k=0}^{N-1} \lambda(x_k) - \sum_{k=0}^{N-1} \lambda(x_{k+1}) - N\, l(x_\mathrm{s}, u_\mathrm{s}) \\
&= \sum_{k=0}^{N-1} l(x_k, u_k) + \underbrace{\lambda(\hat{x}_0) - \lambda(x_\mathrm{s}) - N\, l(x_\mathrm{s}, u_\mathrm{s})}_{\text{constant}}
\end{aligned}
$$

Rotated MPC Problem

Original Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k, u_k), \\
& h(x_k, u_k) \geq 0, \\
& x_N = x_{\mathrm{s}}.
\end{aligned}
$$

Rotated Problem

$$
\begin{aligned}
\bar{V}_N(\hat{x}_0) := \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} L(x_k, u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k, u_k), \\
& h(x_k, u_k) \geq 0, \\
& x_N = x_{\mathrm{s}}.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{N-1} L(x, u) &= \sum_{k=0}^{N-1} l(x_k, u_k) + \sum_{k=0}^{N-1} \lambda(x_k) - \sum_{k=0}^{N-1} \lambda(f(x_k, u_k)) - \sum_{k=0}^{N-1} l(x_{\mathrm{s}}, u_{\mathrm{s}}) \\
&= \sum_{k=0}^{N-1} l(x_k, u_k) + \sum_{k=0}^{N-1} \lambda(x_k) - \sum_{k=0}^{N-1} \lambda(x_{k+1}) - N\, l(x_{\mathrm{s}}, u_{\mathrm{s}}) \\
&= \sum_{k=0}^{N-1} l(x_k, u_k) + \underbrace{\boldsymbol{\lambda(\hat{x}_0)} - \lambda(x_{\mathrm{s}}) - N\, l(x_{\mathrm{s}}, u_{\mathrm{s}})}_{\text{constant}}
\end{aligned}
$$

Rotated MPC Problem

Original Problem

$$
\begin{aligned}
V_N(\hat{x}_0) := \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k, u_k), \\
& h(x_k, u_k) \geq 0, \\
& x_N = x_\mathrm{s}.
\end{aligned}
$$

Rotated Problem

$$
\begin{aligned}
\bar{V}_N(\hat{x}_0) := \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} L(x_k, u_k) \\
\text{s.t.} \quad & x_0 = \hat{x}_0, \\
& x_{k+1} = f(x_k, u_k), \\
& h(x_k, u_k) \geq 0, \\
& x_N = x_\mathrm{s}.
\end{aligned}
$$

$$
\sum_{k=0}^{N-1} L(x, u) = \sum_{k=0}^{N-1} l(x_k, u_k) + \lambda(\hat{x}_0) + \text{constant}
$$

$$
\bar{V}_N(\hat{x}_0) = V_N(\hat{x}_0) + \lambda(\hat{x}_0) + \text{constant}
$$

The two problems deliver the **same primal solution**!

Same stability properties

Stability Assumption

Rotated cost: $L(x, u) = l(x, u) - l(x_{\mathrm{s}}, u_{\mathrm{s}}) + \lambda(x) - \lambda(f(x, u))$

Stability Assumption

Rotated cost: $L(x, u) = l(x, u) - l(x_{\mathrm{s}}, u_{\mathrm{s}}) + \lambda(x) - \lambda(f(x, u))$

What conditions on $\lambda(x)$?

Stability Assumption

Rotated cost: $L(x,u) = l(x,u) - l(x_s,u_s) + \lambda(x) - \lambda(f(x,u))$

What conditions on $\lambda(x)$?

Strict Dissipativity
System strictly dissipative wrt the supply rate $s(x,u)$ if $\exists\ \lambda(x),\ \rho(x) \in \mathcal{K}$ s.t.

$$\lambda(f(x,u)) - \lambda(x) \leq -\rho(\|x - x_s\|) + s(x,u), \qquad \forall\ (x,u) \in \mathbb{Z}$$

We are interested in $s(x,u) = l(x,u) - l(x_s,u_s)$

Stability Assumption

Rotated cost: $L(x,u) = l(x,u) - l(x_\mathrm{s}, u_\mathrm{s}) + \lambda(x) - \lambda(f(x,u))$

What conditions on $\lambda(x)$?

Strict Dissipativity
System strictly dissipative wrt the supply rate $s(x,u)$ if $\exists\ \lambda(x),\ \rho(x) \in \mathcal{K}$ s.t.

$$\lambda(f(x,u)) - \lambda(x) \leq -\rho(\|x - x_\mathrm{s}\|) + s(x,u), \qquad \forall\ (x,u) \in \mathbb{Z}$$

We are interested in $s(x,u) = l(x,u) - l(x_\mathrm{s}, u_\mathrm{s})$

This entails

$$L(x,u) \geq \rho(x - x_\mathrm{s})$$

Stability Assumption

Rotated cost: $L(x,u) = l(x,u) - l(x_{\mathrm{s}},u_{\mathrm{s}}) + \lambda(x) - \lambda(f(x,u))$

What conditions on $\lambda(x)$?

Strict Dissipativity

System strictly dissipative wrt the supply rate $s(x,u)$ if $\exists\ \lambda(x),\ \rho(x) \in \mathcal{K}$ s.t.

$$\lambda(f(x,u)) - \lambda(x) \leq -\rho(\|x - x_{\mathrm{s}}\|) + s(x,u), \qquad \forall\ (x,u) \in \mathbb{Z}$$

We are interested in $s(x,u) = l(x,u) - l(x_{\mathrm{s}},u_{\mathrm{s}})$

This entails

$$L(x,u) \geq \rho(x - x_{\mathrm{s}})$$

This only works for compact constraint sets!! $(x,u) \in \mathbb{Z}$

Different Terminal Conditions

Different Terminal Conditions

Terminal penalty + set constraint

$$\exists\ \mathbb{X}_\mathrm{f},\ \kappa(x)\ \text{s.t.}\ \forall\ x \in \mathbb{X}_\mathrm{f}$$

$$V_\mathrm{f}(f(x,\kappa(x))) \leq V_\mathrm{f}(x) - l(x,\kappa(x)) + l(x_\mathrm{s}, u_\mathrm{s})$$

Different Terminal Conditions

Terminal penalty + set constraint

$$\exists\ \mathbb{X}_{\mathrm{f}},\ \kappa(x)\ \text{s.t.}\ \forall\ x \in \mathbb{X}_{\mathrm{f}}$$

$$V_{\mathrm{f}}(f(x,\kappa(x))) \leq V_{\mathrm{f}}(x) - l(x,\kappa(x)) + l(x_{\mathrm{s}}, u_{\mathrm{s}})$$

With an end point constraint we had:

$$V_N(f(\hat{x}_i, u_0^\star)) \leq V_N(\hat{x}_i) - l(\hat{x}_i, u_0^\star) + \underbrace{\boldsymbol{l(x_{\mathrm{s}}, u_{\mathrm{s}})}}_{=0}$$

Different Terminal Conditions

Terminal penalty + set constraint

$$\exists\ \mathbb{X}_\mathrm{f},\ \kappa(x) \text{ s.t. } \forall\ x \in \mathbb{X}_\mathrm{f}$$

$$V_\mathrm{f}(f(x,\kappa(x))) \le V_\mathrm{f}(x) - l(x,\kappa(x)) + l(x_\mathrm{s},u_\mathrm{s})$$

With an end point constraint we had:

$$V_N(f(\hat{x}_i,u_0^\star)) \le V_N(\hat{x}_i) - l(\hat{x}_i,u_0^\star) + \underbrace{\boldsymbol{l(x_\mathrm{s},u_\mathrm{s})}}_{=0}$$

now we have:

$$V_N(f(\hat{x}_i,u_0^\star)) \le V_N(\hat{x}_i) - l(\hat{x}_i,u_0^\star) + \underbrace{\boldsymbol{V_\mathrm{f}(f(x,\kappa(x))) - V_\mathrm{f}(x) + l(x,\kappa(x))}}_{\le l(x_\mathrm{s},u_\mathrm{s})}$$

Different Terminal Conditions

Terminal penalty + set constraint

$$\exists\ \mathbb{X}_\mathrm{f},\ \kappa(x)\ \text{s.t.}\ \forall\ x \in \mathbb{X}_\mathrm{f}$$
$$V_\mathrm{f}(f(x,\kappa(x))) \leq V_\mathrm{f}(x) - l(x,\kappa(x)) + l(x_\mathrm{s},u_\mathrm{s})$$

With an end point constraint we had:

$$V_N(f(\hat{x}_i,u_0^\star)) \leq V_N(\hat{x}_i) - l(\hat{x}_i,u_0^\star) + \underbrace{\boldsymbol{l(x_\mathrm{s},u_\mathrm{s})}}_{=0}$$

now we have:

$$V_N(f(\hat{x}_i,u_0^\star)) \leq V_N(\hat{x}_i) - l(\hat{x}_i,u_0^\star) + \underbrace{\boldsymbol{V_\mathrm{f}(f(x,\kappa(x))) - V_\mathrm{f}(x) + l(x,\kappa(x))}}_{\leq l(x_\mathrm{s},u_\mathrm{s})}$$

rotating:

$$\bar{V}_N(f(\hat{x}_i,u_0^\star)) \leq \bar{V}_N(\hat{x}_i) - L(\hat{x}_i,u_0^\star) + \underbrace{\boldsymbol{\bar{V}_\mathrm{f}(f(x,\kappa(x))) - \bar{V}_\mathrm{f}(x) + L(x,\kappa(x))}}_{\leq 0}$$

with:

$$\bar{V}_\mathrm{f}(x) = V_\mathrm{f}(x) + \lambda(x) - V_\mathrm{f}(x_\mathrm{s}) - \lambda(x_\mathrm{s})$$

Different Terminal Conditions

No terminal penalty nor constraints

Different Terminal Conditions

No terminal penalty nor constraints

- Technical assumptions + horizon long enough [Grüne, Automatica2013]
  $\rightarrow$ **practical stability** to $\mathcal{N}_\rho(x_\mathrm{s})$

Different Terminal Conditions

No terminal penalty nor constraints

- Technical assumptions + horizon long enough [Grüne, Automatica2013]
  $\rightarrow$ **practical stability** to $\mathcal{N}_\rho(x_\mathrm{s})$
- $\lim_{N\rightarrow\infty} \rho = 0$

Different Terminal Conditions

No terminal penalty nor constraints

- Technical assumptions + horizon long enough [Grüne, Automatica2013]
  $\rightarrow$ **practical stability** to $\mathcal{N}_\rho(x_\mathrm{s})$
- $\lim_{N\rightarrow\infty} \rho = 0$
- If $\lambda_\mathrm{s} \neq 0$, $N < \infty$ then $\lim\limits_{k\rightarrow\infty} x_k \neq x_\mathrm{s}$ [Zanon, Faulwasser, JPC2018]

Different Terminal Conditions

No terminal penalty nor constraints

- Technical assumptions + horizon long enough [Grüne, Automatica2013]
  $\rightarrow$ **practical stability** to $\mathcal{N}_\rho(x_\mathrm{s})$
- $\lim_{N\to\infty} \rho = 0$
- If $\lambda_\mathrm{s} \neq 0$, $N < \infty$ then $\lim\limits_{k\to\infty} x_k \neq x_\mathrm{s}$ [Zanon, Faulwasser, JPC2018]
- $\Rightarrow$ **tracking MPC performs better than economic MPC**

Wrap-up

EMPC stability if:

Wrap-up

EMPC stability if:

1. Strict dissipativity

$$\lambda(f(x,u)) - \lambda(x) \leq -\rho(x - x_{\mathrm{s}}) + l(x,u) - l(x_{\mathrm{s}}, u_{\mathrm{s}})$$

Wrap-up

EMPC stability if:

1. Strict dissipativity

$$\lambda(f(x,u)) - \lambda(x) \leq -\rho(x - x_\mathrm{s}) + l(x,u) - l(x_\mathrm{s}, u_\mathrm{s})$$

2. Compact constraint set

$$(x,u) \in \mathbb{Z}$$

Wrap-up

EMPC stability if:

1. Strict dissipativity

$$\lambda(f(x,u)) - \lambda(x) \leq -\rho(x - x_\mathrm{s}) + l(x,u) - l(x_\mathrm{s}, u_\mathrm{s})$$

2. Compact constraint set

$$(x,u) \in \mathbb{Z}$$

3. Suitable terminal conditions
   - Point constraint
   - Set constraint + terminal cost
   - Technical assumptions + long horizon (only practical stability)

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x, u) = u^2$$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x, u) = u^2$$

- Optimal (LQR) feedback: $u = 0$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = u^2$$

- Optimal (LQR) feedback: $u = 0$
- **Unstable!**

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = u^2$$

- Optimal (LQR) feedback: $u = 0$
- **Unstable!**
- Add constraint $\lim_{k\to\infty} x_k = 0$ OR terminal cost $V_\mathrm{f}(x) = ax^2, \ a > 0$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x,u) = u^2$$

- Optimal (LQR) feedback: $u = 0$
- **Unstable!**
- Add constraint $\lim_{k\to\infty} x_k = 0$ OR terminal cost $V_\mathrm{f}(x) = ax^2, \ \ a > 0$
- Optimal (LQR) feedback: $u = -1.5x$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x,u) = u^2$$

- Optimal (LQR) feedback: $u = 0$
- **Unstable!**
- Add constraint $\lim_{k\to\infty} x_k = 0$ OR terminal cost $V_f(x) = ax^2, \;\; a > 0$
- Optimal (LQR) feedback: $u = -1.5x$
- $V(x) = 3x^2$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x,u) = u^2$$

- Optimal (LQR) feedback: $u = 0$
- **Unstable!**
- Add constraint $\lim_{k\to\infty} x_k = 0$ OR terminal cost $V_\mathrm{f}(x) = ax^2, \ \ a > 0$
- Optimal (LQR) feedback: $u = -1.5x$
- $V(x) = 3x^2$
- Riccati equation:

$$P = 4P - \frac{4P^2}{1+P} \qquad 3P - P^2 = 0 \qquad P \in \{0, 3\}$$

$$K \in \{0, 1.5\}$$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x,u) = u^2$$

- Optimal (LQR) feedback: $u = 0$
- **Unstable!**
- Add constraint $\lim_{k\to\infty} x_k = 0$ OR terminal cost $V_\mathrm{f}(x) = ax^2, \ \ a > 0$
- Optimal (LQR) feedback: $u = -1.5x$
- $V(x) = 3x^2$
- Riccati equation:

$$P = 4P - \frac{4P^2}{1+P} \qquad 3P - P^2 = 0 \qquad P \in \{0, 3\}$$

$$K \in \{0, 1.5\}$$

- $\exists\ \lambda(x) = -0.5x^2$ s.t. $L(x,u)$ pos. def.

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x, u) = u^2$$

- Add constraint $|x_k| \leq 1$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x, u) = u^2$$

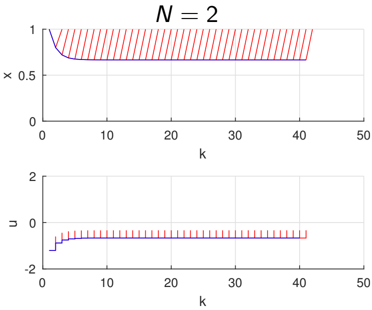
- Add constraint $|x_k| \leq 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x,u) = u^2$$

- Add constraint $|x_k| \leq 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

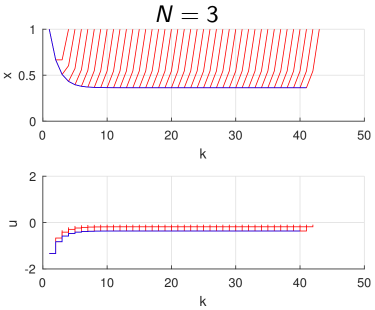


A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x,u) = u^2$$

- Add constraint $|x_k| \leq 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

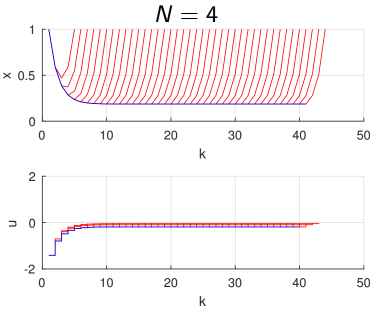


A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = u^2$$

- Add constraint $|x_k| \le 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

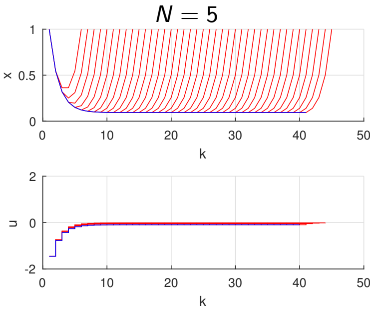


A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x,u) = u^2$$

- Add constraint $|x_k| \leq 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

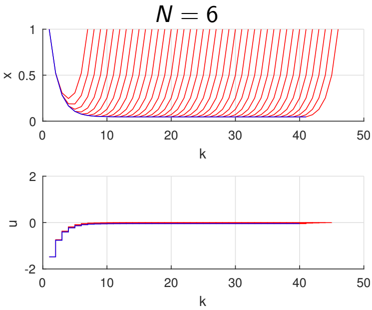


A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_{+} = 2x + u, \qquad l(x,u) = u^2$$

- Add constraint $|x_k| \leq 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

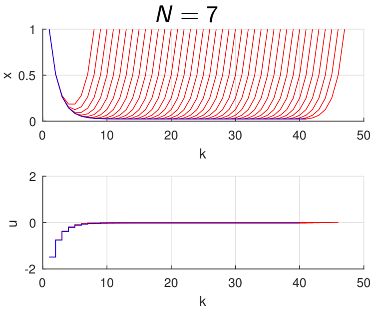


A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = u^2$$

- Add constraint $|x_k| \leq 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

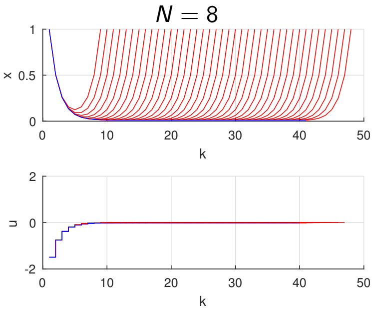


A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = u^2$$

- Add constraint $|x_k| \leq 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

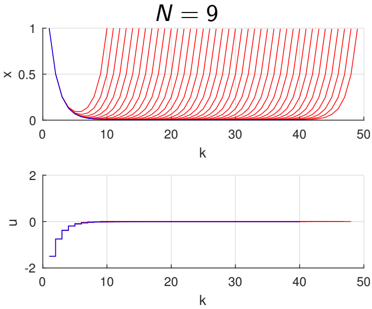


A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = u^2$$

- Add constraint $|x_k| \leq 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

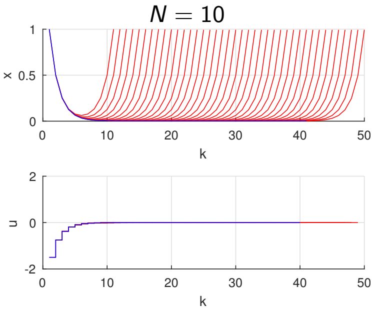


A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x,u) = u^2$$

- Add constraint $|x_k| \le 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$

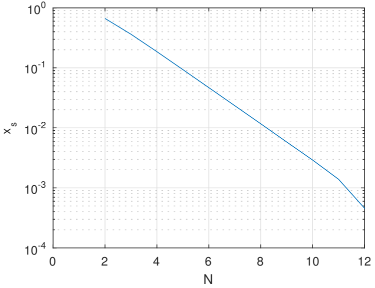


A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = u^2$$

- Add constraint $|x_k| \leq 1$
- With increasing N we stabilise the system closer to 0, but only get there for $N = \infty$



A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x, u) = x^2 + u^2 - x - u$$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = x^2 + u^2 - x - u$$

$$0 = \min_{x,u} x^2 + u^2 - x - u \qquad \text{s.t.} \qquad x = 2x + u$$

A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = x^2 + u^2 - x - u$$

$$0 = \min_{x,u} x^2 + u^2 - x - u \qquad \text{s.t.} \qquad x = 2x + u$$

$$x_\mathrm{s} = 0, \qquad u_\mathrm{s} = 0, \qquad \lambda_\mathrm{s} = 1$$

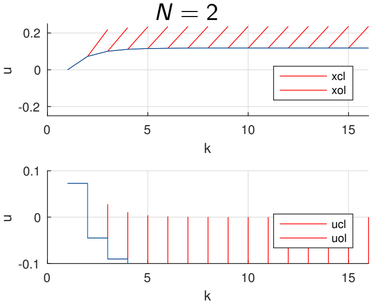
A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = x^2 + u^2 - x - u$$

$$0 = \min_{x,u} x^2 + u^2 - x - u \qquad \text{s.t.} \qquad x = 2x + u$$

$$x_\mathrm{s} = 0, \qquad u_\mathrm{s} = 0, \qquad \lambda_\mathrm{s} = 1$$



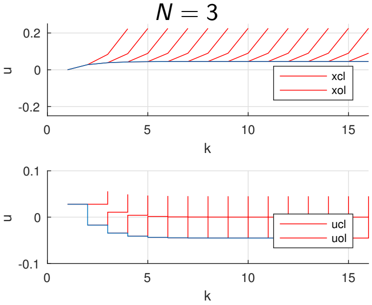
A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = x^2 + u^2 - x - u$$

$$0 = \min_{x,u} x^2 + u^2 - x - u \qquad \text{s.t.} \qquad x = 2x + u$$

$$x_s = 0, \qquad u_s = 0, \qquad \lambda_s = 1$$



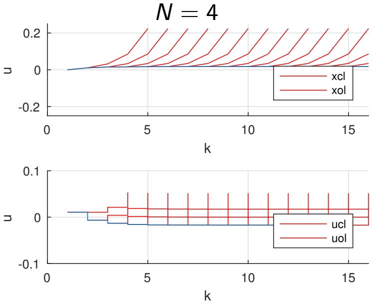
A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = x^2 + u^2 - x - u$$

$$0 = \min_{x,u} x^2 + u^2 - x - u \qquad \text{s.t.} \qquad x = 2x + u$$

$$x_\text{s} = 0, \qquad u_\text{s} = 0, \qquad \lambda_\text{s} = 1$$



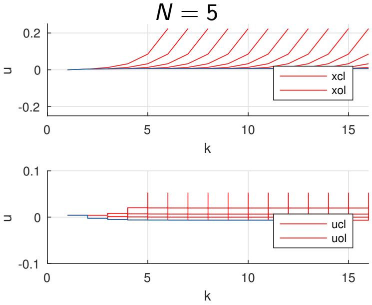
A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = x^2 + u^2 - x - u$$

$$0 = \min_{x,u} x^2 + u^2 - x - u \qquad \text{s.t.} \qquad x = 2x + u$$

$$x_\mathrm{s} = 0, \qquad u_\mathrm{s} = 0, \qquad \lambda_\mathrm{s} = 1$$



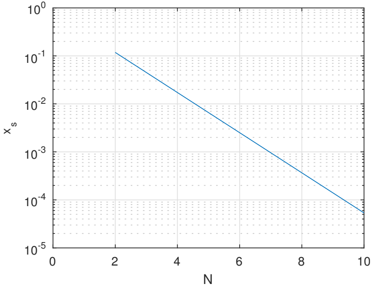
A Simple (but Tricky) Economic MPC Example

- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = x^2 + u^2 - x - u$$

$$0 = \min_{x,u} x^2 + u^2 - x - u \qquad \text{s.t.} \qquad x = 2x + u$$

$$x_\mathrm{s} = 0, \qquad u_\mathrm{s} = 0, \qquad \lambda_\mathrm{s} = 1$$



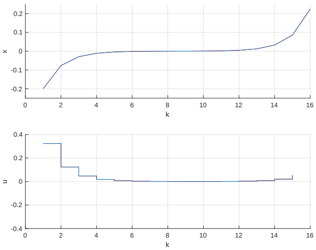
A Simple (but Tricky) Economic MPC Example

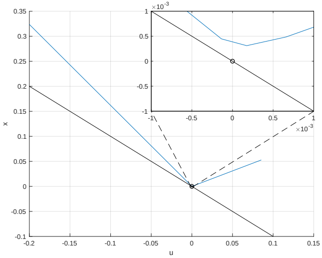
- Consider the simple system:

$$x_+ = 2x + u, \qquad\qquad l(x,u) = x^2 + u^2 - x - u$$

$$0 = \min_{x,u} x^2 + u^2 - x - u \qquad \text{s.t.} \qquad x = 2x + u$$

$$x_\mathrm{s} = 0, \qquad u_\mathrm{s} = 0, \qquad \lambda_\mathrm{s} = 1$$





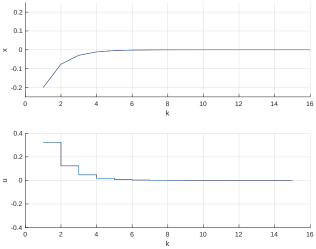
A Simple (but Tricky) Economic MPC Example

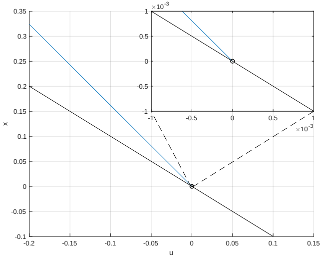
- Consider the simple system:

$$x_+ = 2x + u, \qquad l(x,u) = x^2 + u^2 - x - u$$

$$0 = \min_{x,u} x^2 + u^2 \qquad \text{s.t.} \qquad x = 2x + u$$

$$x_\mathrm{s} = 0, \qquad u_\mathrm{s} = 0, \qquad \lambda_\mathrm{s} = 1 \qquad \lambda(x) = -x$$

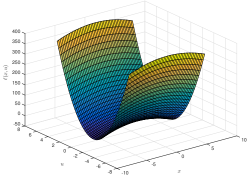




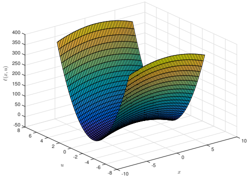
A Linear Economic MPC Example

- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2$$



A Linear Economic MPC Example



- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2$$

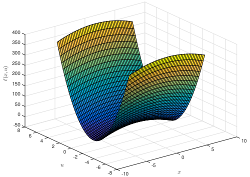
- Optimal (LQR) feedback: $u = 0.0113x$

A Linear Economic MPC Example

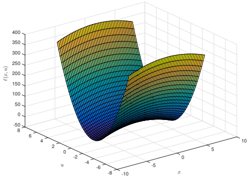
- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2$$

- Optimal (LQR) feedback: $u = 0.0113x$
- Then $x_+ = 0.1113x \;\Rightarrow$ stability



A Linear Economic MPC Example

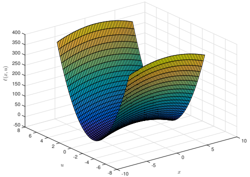


- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2$$

- Optimal (LQR) feedback: $u = 0.0113x$
- Then $x_+ = 0.1113x \ \Rightarrow$ stability
- Cost-to-go: $V(x) = -1.0113x^2$

A Linear Economic MPC Example



- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2$$

- Optimal (LQR) feedback: $u = 0.0113x$
- Then $x_+ = 0.1113x \Rightarrow$ stability
- Cost-to-go: $V(x) = -1.0113x^2$

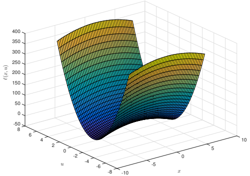
- Rotate using $\lambda(x) = 5.475x^2$

A Linear Economic MPC Example

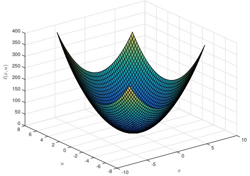
- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2$$

- Optimal (LQR) feedback: $u = 0.0113x$
- Then $x_+ = 0.1113x \;\Rightarrow$ stability
- Cost-to-go: $V(x) = -1.0113x^2$



- Rotate using $\lambda(x) = 5.475x^2$
- Rotated the stage cost
  $L(x,u) = 4.4204x^2 - 1.095ux + 4.5249u^2$

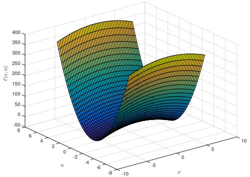


A Linear Economic MPC Example

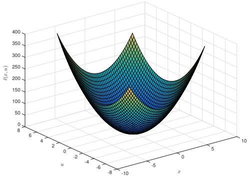
- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2$$

- Optimal (LQR) feedback: $u = 0.0113x$
- Then $x_+ = 0.1113x \;\Rightarrow$ stability
- Cost-to-go: $V(x) = -1.0113x^2$



- Rotate using $\lambda(x) = 5.475x^2$
- Rotated the stage cost
  $L(x,u) = 4.4204x^2 - 1.095ux + 4.5249u^2$
- We obtain $u = 0.0113x$ and $\bar{V}(x) = 4.4639x^2$



A Non-Steady State MPC Example

Continuously Stirred Tank Reactor: $A \rightarrow B$

A Non-Steady State MPC Example

Continuously Stirred Tank Reactor: $A \to B$

- States $x = (c_\mathrm{A}, c_\mathrm{B})$, control $u = q$ (flow). Sampling: $t_\mathrm{s} = 0.5\ \mathrm{min}$

$$\begin{aligned}
\dot{c}_\mathrm{A} &= \frac{q}{10}(1 - c_\mathrm{A}) - 0.4 c_\mathrm{A} \qquad & (c_\mathrm{A}, c_\mathrm{B}) \in [0,1]^2 \\
\dot{c}_\mathrm{B} &= -\frac{q}{10} c_\mathrm{B} + 0.4 c_\mathrm{A} \qquad & q \in [0, 20]
\end{aligned}$$

A Non-Steady State MPC Example

Continuously Stirred Tank Reactor: $A \rightarrow B$

- States $x = (c_\mathrm{A}, c_\mathrm{B})$, control $u = q$ (flow). Sampling: $t_\mathrm{s} = 0.5\ \mathrm{min}$

$$\begin{aligned} \dot{c}_\mathrm{A} &= \frac{q}{10}(1 - c_\mathrm{A}) - 0.4 c_\mathrm{A} && (c_\mathrm{A}, c_\mathrm{B}) \in [0,1]^2 \\ \dot{c}_\mathrm{B} &= -\frac{q}{10} c_\mathrm{B} + 0.4 c_\mathrm{A} && q \in [0, 20] \end{aligned}$$

- Stage cost: $l(x, u) = 2 q c_\mathrm{A} - 1.5 q$

A Non-Steady State MPC Example

Continuously Stirred Tank Reactor: $A \to B$

- States $x = (c_\mathrm{A}, c_\mathrm{B})$, control $u = q$ (flow). Sampling: $t_\mathrm{s} = 0.5\ \mathrm{min}$

$$\begin{aligned} \dot{c}_\mathrm{A} &= \frac{q}{10}(1 - c_\mathrm{A}) - 0.4 c_\mathrm{A} & \qquad (c_\mathrm{A}, c_\mathrm{B}) \in [0,1]^2 \\ \dot{c}_\mathrm{B} &= -\frac{q}{10} c_\mathrm{B} + 0.4 c_\mathrm{A} & \qquad q \in [0, 20] \end{aligned}$$

- Stage cost: $l(x, u) = 2 q c_\mathrm{A} - 1.5 q$
- Optimal Steady State: $x_\mathrm{s} = (0.5, 0.5)\ \mathrm{mol/l}$, $u_\mathrm{s} = 4\ \mathrm{l/(mol\ min)}$

A Non-Steady State MPC Example

Continuously Stirred Tank Reactor: $A \to B$

- States $x = (c_\mathrm{A}, c_\mathrm{B})$, control $u = q$ (flow). Sampling: $t_\mathrm{s} = 0.5\ \mathrm{min}$

$$\dot{c}_\mathrm{A} = \frac{q}{10}(1 - c_\mathrm{A}) - 0.4c_\mathrm{A} \qquad (c_\mathrm{A}, c_\mathrm{B}) \in [0,1]^2$$

$$\dot{c}_\mathrm{B} = -\frac{q}{10}c_\mathrm{B} + 0.4c_\mathrm{A} \qquad q \in [0, 20]$$

- Stage cost: $l(x,u) = 2qc_\mathrm{A} - 1.5q$
- Optimal Steady State: $x_\mathrm{s} = (0.5, 0.5)\ \mathrm{mol/l}$, $u_\mathrm{s} = 4\ \mathrm{l/(mol\ min)}$
- $l(x_\mathrm{s}, u_\mathrm{s}) = -2$,

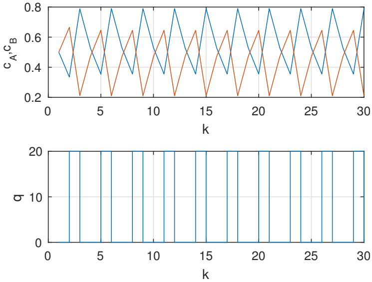
A Non-Steady State MPC Example

Continuously Stirred Tank Reactor: $A \to B$

- States $x = (c_{\mathrm{A}}, c_{\mathrm{B}})$, control $u = q$ (flow). Sampling: $t_{\mathrm{s}} = 0.5\ \mathrm{min}$

$$\begin{aligned} \dot{c}_{\mathrm{A}} &= \frac{q}{10}(1 - c_{\mathrm{A}}) - 0.4c_{\mathrm{A}} & (c_{\mathrm{A}}, c_{\mathrm{B}}) \in [0,1]^2 \\ \dot{c}_{\mathrm{B}} &= -\frac{q}{10}c_{\mathrm{B}} + 0.4c_{\mathrm{A}} & q \in [0,20] \end{aligned}$$

- Stage cost: $l(x,u) = 2qc_{\mathrm{A}} - 1.5q$
- Optimal Steady State: $x_{\mathrm{s}} = (0.5, 0.5)\ \mathrm{mol/l}$, $u_{\mathrm{s}} = 4\ \mathrm{l/(mol\ min)}$
- $l(x_{\mathrm{s}}, u_{\mathrm{s}}) = -2$, $\quad \mathrm{avg}_k\, l(x_k^{\mathrm{p}}, u_k^{\mathrm{p}}) \approx -5.3$



Sketch of the Idea [Zanon, Gros, DIehl, JPC2014, JPC2016, IFAC2017]

Sketch of the Idea [Zanon, Gros, DIehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\text{tracking}}(\hat{x}_0) - u^{\text{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\text{s}}\|^2)$$

Sketch of the Idea [Zanon, Gros, DIehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\mathrm{tracking}}(\hat{x}_0) - u^{\mathrm{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\mathrm{s}}\|^2)$$

ENMPC $\leftrightarrow$ LMPC

Sketch of the Idea [Zanon, Gros, DIehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\mathrm{tracking}}(\hat{x}_0) - u^{\mathrm{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\mathrm{s}}\|^2)$$

ENMPC $\leftrightarrow$ LMPC

Solve the optimal steady-state problem: $x_{\mathrm{s}}, u_{\mathrm{s}}$

Sketch of the Idea [Zanon, Gros, DIehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\text{tracking}}(\hat{x}_0) - u^{\text{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\rm s}\|^2)$$

ENMPC $\leftrightarrow$ LMPC

Solve the optimal steady-state problem: $x_{\rm s}, u_{\rm s}$
Stage cost: Hessian $= \nabla^2 \mathcal{L}(x_{\rm s}, u_{\rm s}, \lambda_{\rm s})$, gradient $= \nabla \ell(x_{\rm s}, u_{\rm s})$

Sketch of the Idea [Zanon, Gros, Diehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\text{tracking}}(\hat{x}_0) - u^{\text{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\mathrm{s}}\|^2)$$

ENMPC $\leftrightarrow$ **E**LMPC

Solve the optimal steady-state problem: $x_{\mathrm{s}}, u_{\mathrm{s}}$
Stage cost: Hessian $= \nabla^2 \mathcal{L}(x_{\mathrm{s}}, u_{\mathrm{s}}, \lambda_{\mathrm{s}})$, gradient $= \nabla \ell(x_{\mathrm{s}}, u_{\mathrm{s}})$

Lagrangian Hessian is typically indefinite!

Locally Equivalent to Economic MPC

Sketch of the Idea [Zanon, Gros, DIehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\text{tracking}}(\hat{x}_0) - u^{\text{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_\mathrm{s}\|^2)$$

ENMPC $\leftrightarrow$ ELMPC

Solve the optimal steady-state problem: $x_\mathrm{s}, u_\mathrm{s}$
Stage cost: Hessian $= \nabla^2 \mathcal{L}(x_\mathrm{s}, u_\mathrm{s}, \lambda_\mathrm{s})$, gradient $= \nabla \ell(x_\mathrm{s}, u_\mathrm{s})$

Lagrangian Hessian is typically indefinite!

Can we design a positive definite LMPC?

Locally Equivalent to Economic MPC

Sketch of the Idea [Zanon, Gros, Diehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\mathrm{tracking}}(\hat{x}_0) - u^{\mathrm{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\mathrm{s}}\|^2)$$

ENMPC $\leftrightarrow$ ELMPC $\leftrightarrow$ PD LMPC

Solve the optimal steady-state problem: $x_{\mathrm{s}}, u_{\mathrm{s}}$
Stage cost: Hessian $= \nabla^2 \mathcal{L}(x_{\mathrm{s}}, u_{\mathrm{s}}, \lambda_{\mathrm{s}})$, gradient $= \nabla \ell(x_{\mathrm{s}}, u_{\mathrm{s}})$

Lagrangian Hessian is typically indefinite!

Can we design a positive definite LMPC?
Yes! Provided that EMPC is locally stabilising

Locally Equivalent to Economic MPC

Sketch of the Idea [Zanon, Gros, DIehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\mathrm{tracking}}(\hat{x}_0) - u^{\mathrm{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\mathrm{s}}\|^2)$$

ENMPC $\leftrightarrow$ ELMPC $\leftrightarrow$ PD LMPC

Solve the optimal steady-state problem: $x_{\mathrm{s}}, u_{\mathrm{s}}$
Stage cost: Hessian $= \nabla^2 \mathcal{L}(x_{\mathrm{s}}, u_{\mathrm{s}}, \lambda_{\mathrm{s}})$, gradient $= \nabla \ell(x_{\mathrm{s}}, u_{\mathrm{s}})$

Lagrangian Hessian is typically indefinite!

Can we design a positive definite LMPC?
Yes! Provided that EMPC is locally stabilising

How do we compute it?

Locally Equivalent to Economic MPC

Sketch of the Idea [Zanon, Gros, Dlehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\text{tracking}}(\hat{x}_0) - u^{\text{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\mathrm{s}}\|^2)$$

ENMPC $\leftrightarrow$ ELMPC $\leftrightarrow$ PD LMPC

Solve the optimal steady-state problem: $x_{\mathrm{s}}, u_{\mathrm{s}}$
Stage cost: Hessian $= \nabla^2 \mathcal{L}(x_{\mathrm{s}}, u_{\mathrm{s}}, \lambda_{\mathrm{s}})$, gradient $= \nabla \ell(x_{\mathrm{s}}, u_{\mathrm{s}})$

Lagrangian Hessian is typically indefinite!

Can we design a positive definite LMPC?
Yes! Provided that EMPC is locally stabilising

How do we compute it? **Solve one SDP offline**

Locally Equivalent to Economic MPC

Sketch of the Idea [Zanon, Gros, Diehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\text{tracking}}(\hat{x}_0) - u^{\text{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\text{s}}\|^2)$$

$$\text{ENMPC} \leftrightarrow \text{ELMPC} \leftrightarrow \text{PD LMPC} \leftrightarrow \text{PD tracking NMPC}$$

Solve the optimal steady-state problem: $x_{\text{s}}, u_{\text{s}}$
Stage cost: Hessian $= \nabla^2 \mathcal{L}(x_{\text{s}}, u_{\text{s}}, \lambda_{\text{s}})$, gradient $= \nabla \ell(x_{\text{s}}, u_{\text{s}})$

Lagrangian Hessian is typically indefinite!

Can we design a positive definite LMPC?
Yes! Provided that EMPC is locally stabilising

How do we compute it? **Solve one SDP offline**

What about NMPC?

Locally Equivalent to Economic MPC

Sketch of the Idea [Zanon, Gros, Diehl, JPC2014, JPC2016, IFAC2017]

Design a quadratic stage cost such that:

$$\|u^{\mathrm{tracking}}(\hat{x}_0) - u^{\mathrm{economic}}(\hat{x}_0)\| = O(\|\hat{x}_0 - x_{\mathrm{s}}\|^2)$$

ENMPC $\leftrightarrow$ ELMPC $\leftrightarrow$ PD LMPC $\leftrightarrow$ PD tracking NMPC

Solve the optimal steady-state problem: $x_{\mathrm{s}}, u_{\mathrm{s}}$
Stage cost: Hessian $= \nabla^2\mathcal{L}(x_{\mathrm{s}}, u_{\mathrm{s}}, \lambda_{\mathrm{s}})$, gradient $= \nabla\ell(x_{\mathrm{s}}, u_{\mathrm{s}})$

Lagrangian Hessian is typically indefinite!

Can we design a positive definite LMPC?
Yes! Provided that EMPC is locally stabilising

How do we compute it? **Solve one SDP offline**

What about NMPC? **We need to use a trick!**

ENMPC $\leftrightarrow$ ELMPC

ENMPC NLP: $\min_w \quad \bar{f}(w,t) \quad \text{s.t.} \quad \bar{g}(w,t) = 0, \quad \bar{h}(w,t) \geq 0,$

with $v^*(t) = (w^*(t), \lambda^*(t), \mu^*(t))$

ENMPC ↔ ELMPC

ENMPC NLP: $\min_w \quad \bar{f}(w,t) \quad \text{s.t.} \quad \bar{g}(w,t) = 0, \quad \bar{h}(w,t) \geq 0,$

$$\text{with } v^*(t) = (w^*(t), \lambda^*(t), \mu^*(t))$$

$$\frac{\partial v^*}{\partial t}$$

ENMPC $\leftrightarrow$ ELMPC

ENMPC NLP: $\min_w \quad \bar{f}(w,t) \quad \text{s.t.} \quad \bar{g}(w,t)=0, \quad \bar{h}(w,t)\geq 0,$

with $v^*(t) = (w^*(t), \lambda^*(t), \mu^*(t))$

$$\frac{\partial v^*}{\partial t} = \delta v^*$$

$$\begin{aligned}\min_{\delta w} \quad & \frac{1}{2}\delta w^\top \nabla_w^2 \bar{\mathcal{L}} \delta w + \left(\frac{\partial}{\partial t}\nabla_w \bar{\mathcal{L}}\right)^\top \delta w \\ \text{s.t.} \quad & \nabla_t \bar{g} + \nabla_w \bar{g}^\top \delta w = 0, \\ & \nabla_t \bar{h} + \nabla_w \bar{h}^\top \delta w \geq 0.\end{aligned}$$

ENMPC $\leftrightarrow$ ELMPC

ENMPC NLP: $\min_w \quad \bar{f}(w,t) \quad \text{s.t.} \quad \bar{g}(w,t) = 0, \quad \bar{h}(w,t) \geq 0,$

with $v^*(t) = (w^*(t), \lambda^*(t), \mu^*(t))$

$$\frac{\partial v^*}{\partial t} = \delta v^* \qquad \begin{aligned} \min_{\delta w} \quad & \frac{1}{2}\delta w^\top \nabla_w^2 \bar{\mathcal{L}} \delta w + \left(\frac{\partial}{\partial t}\nabla_w \bar{\mathcal{L}}\right)^\top \delta w \\ \text{s.t.} \quad & \nabla_t \bar{g} + \nabla_w \bar{g}^\top \delta w = 0, \\ & \nabla_t \bar{h} + \nabla_w \bar{h}^\top \delta w \geq 0. \end{aligned}$$

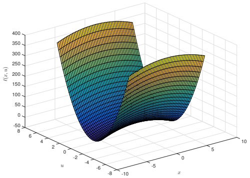
ELMPC QP:

$$\begin{aligned} \min_{w} \quad & \frac{1}{2} w^\top \nabla_w^2 \bar{\mathcal{L}} w + \nabla_w \bar{f}^\top w \\ \text{s.t.} \quad & \bar{g}(w^*,t) + \nabla_w \bar{g}^\top w = 0, \\ & \bar{h}(w^*,t) + \nabla_w \bar{h}^\top w \geq 0, \end{aligned}$$

ELMPC ↔ PD LMPC

- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2.$$

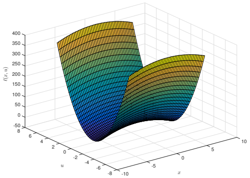


ELMPC ↔ PD LMPC

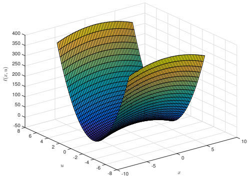
- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2.$$

- Optimal (LQR) feedback: $u = 0.0113x$.



ELMPC ↔ PD LMPC

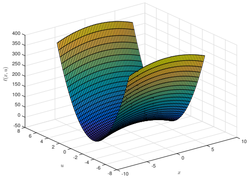


- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2.$$

- Optimal (LQR) feedback: $u = 0.0113x$.
- Then $x_+ = 0.1113x \;\Rightarrow$ stability.

ELMPC ↔ PD LMPC



- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2.$$

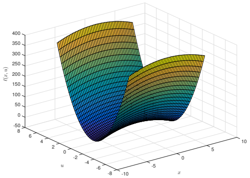
- Optimal (LQR) feedback: $u = 0.0113x$.
- Then $x_+ = 0.1113x \;\Rightarrow$ stability.
- Cost-to-go: $V(x) = -1.0113x^2$.

ELMPC ↔ PD LMPC

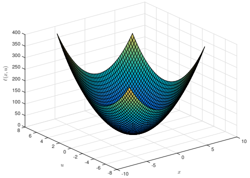
- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2.$$

- Optimal (LQR) feedback: $u = 0.0113x$.
- Then $x_+ = 0.1113x \;\Rightarrow$ stability.
- Cost-to-go: $V(x) = -1.0113x^2$.



- Consider the stage cost $l(x,u) = 4.4204x^2 - 1.095ux + 4.5249u^2$

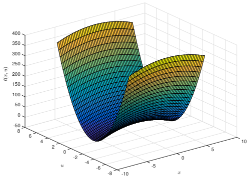


ELMPC ↔ PD LMPC

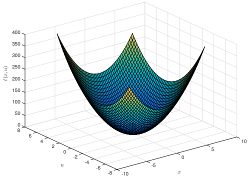
- Consider the simple system:

$$x_+ = 0.1x + u, \qquad l(x,u) = -x^2 + 10u^2.$$

- Optimal (LQR) feedback: $u = 0.0113x$.
- Then $x_+ = 0.1113x \Rightarrow$ stability.
- Cost-to-go: $V(x) = -1.0113x^2$.



- Consider the stage cost $l(x,u) = 4.4204x^2 - 1.095ux + 4.5249u^2$
- We obtain $u = 0.0113x$ and $V(x) = 4.4639x^2$!



ELMPC $\leftrightarrow$ PD LMPC

How to compute that?

ELMPC $\leftrightarrow$ PD LMPC

How to compute that? Quadratic cost rotation

ELMPC ↔ PD LMPC

How to compute that? Quadratic cost rotation

$$l(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\top \underbrace{\nabla^2 \mathcal{L}(x_\mathrm{s}, u_\mathrm{s}, \lambda_\mathrm{s})}_{:=H} \begin{bmatrix} x \\ u \end{bmatrix}, \qquad x^+ = Ax + Bu,$$

ELMPC $\leftrightarrow$ PD LMPC

How to compute that? Quadratic cost rotation

$$l(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\top \underbrace{\nabla^2 \mathcal{L}(x_\mathrm{s}, u_\mathrm{s}, \lambda_\mathrm{s})}_{:=H} \begin{bmatrix} x \\ u \end{bmatrix}, \qquad x^+ = Ax + Bu,$$

Quadratic rotation operator:

$$\mathcal{H}(\delta P) = \begin{bmatrix} A^\top \delta P A - \delta P & A^\top \delta P B \\ B^\top \delta P A & B^\top \delta P B \end{bmatrix}$$

$$L(x,u) = \underbrace{l(x,u) - l(x_\mathrm{s}, u_\mathrm{s})}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top H \begin{bmatrix} x \\ u \end{bmatrix}} + \underbrace{\lambda(x) - \lambda(f(x,u))}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top \mathcal{H}(\delta P) \begin{bmatrix} x \\ u \end{bmatrix}}$$

ELMPC ↔ PD LMPC

How to compute that? Quadratic cost rotation

$$l(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\top \underbrace{\nabla^2 \mathcal{L}(x_\mathrm{s}, u_\mathrm{s}, \lambda_\mathrm{s})}_{:=H} \begin{bmatrix} x \\ u \end{bmatrix}, \qquad x^+ = Ax + Bu,$$

Quadratic rotation operator:

$$\mathcal{H}(\delta P) = \begin{bmatrix} A^\top \delta P A - \delta P & A^\top \delta P B \\ B^\top \delta P A & B^\top \delta P B \end{bmatrix}$$

$$L(x,u) = \underbrace{l(x,u) - l(x_\mathrm{s}, u_\mathrm{s})}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top H \begin{bmatrix} x \\ u \end{bmatrix}} + \underbrace{\lambda(x) - \lambda(f(x,u))}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top \mathcal{H}(\delta P) \begin{bmatrix} x \\ u \end{bmatrix}}$$

Solve the following **convex** SDP

$$\begin{aligned} \min_{\delta P, \alpha, \beta} \quad & \gamma\beta - \alpha \\ \text{s.t.} \quad & H + \mathcal{H}(\delta P) \succeq \alpha I, \\ & H + \mathcal{H}(\delta P) \preceq \beta I, \\ & \alpha \geq \epsilon. \end{aligned}$$

ELMPC ↔ PD LMPC

How to compute that? Quadratic cost rotation

$$l(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\top \underbrace{\nabla^2 \mathcal{L}(x_\mathrm{s}, u_\mathrm{s}, \lambda_\mathrm{s})}_{:=H} \begin{bmatrix} x \\ u \end{bmatrix}, \qquad x^+ = Ax + Bu,$$

Quadratic rotation operator:

$$\mathcal{H}(\delta P) = \begin{bmatrix} A^\top \delta P A - \delta P & A^\top \delta P B \\ B^\top \delta P A & B^\top \delta P B \end{bmatrix}$$

$$L(x,u) = \underbrace{l(x,u) - l(x_\mathrm{s}, u_\mathrm{s})}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top H \begin{bmatrix} x \\ u \end{bmatrix}} + \underbrace{\lambda(x) - \lambda(f(x,u))}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top \mathcal{H}(\delta P) \begin{bmatrix} x \\ u \end{bmatrix}}$$

Solve the following **convex** SDP

$$\begin{aligned} \min_{\delta P, \alpha, \beta} \quad & \gamma\beta - \alpha \\ \text{s.t.} \quad & H + \mathcal{H}(\delta P) \succeq \alpha I, \\ & H + \mathcal{H}(\delta P) \preceq \beta I, \\ & \alpha \geq \epsilon. \end{aligned}$$

ELMPC stabilising ⇒ ∃ SDP solution

ELMPC ↔ PD LMPC

How to compute that? Quadratic cost rotation

$$l(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\top \underbrace{\nabla^2 \mathcal{L}(x_\mathrm{s}, u_\mathrm{s}, \lambda_\mathrm{s})}_{:=H} \begin{bmatrix} x \\ u \end{bmatrix}, \qquad x^+ = Ax + Bu, \qquad C_{\mathbb{A}_\mathrm{s}} \begin{bmatrix} x \\ u \end{bmatrix} \geq 0$$

Quadratic rotation operator:

$$\mathcal{H}(\delta P) = \begin{bmatrix} A^\top \delta P A - \delta P & A^\top \delta P B \\ B^\top \delta P A & B^\top \delta P B \end{bmatrix}$$

$$L(x,u) = \underbrace{l(x,u) - l(x_\mathrm{s}, u_\mathrm{s})}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top H \begin{bmatrix} x \\ u \end{bmatrix}} + \underbrace{\lambda(x) - \lambda(f(x,u))}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top \mathcal{H}(\delta P) \begin{bmatrix} x \\ u \end{bmatrix}}$$

Solve the following **convex** SDP

$$\begin{aligned} \min_{\delta P, \alpha, \beta} \quad & \gamma\beta - \alpha \\ \text{s.t.} \quad & H + \mathcal{H}(\delta P) \succeq \alpha I, \\ & H + \mathcal{H}(\delta P) \preceq \beta I, \\ & \alpha \geq \epsilon. \end{aligned}$$

ELMPC stabilising $\Rightarrow \exists$ SDP solution

ELMPC ↔ PD LMPC

How to compute that? Quadratic cost rotation

$$l(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\top \underbrace{\nabla^2 \mathcal{L}(x_\mathrm{s}, u_\mathrm{s}, \lambda_\mathrm{s})}_{:=H} \begin{bmatrix} x \\ u \end{bmatrix}, \qquad x^+ = Ax + Bu, \qquad C_{\mathbb{A}_\mathrm{s}} \begin{bmatrix} x \\ u \end{bmatrix} \geq 0$$

Quadratic rotation operator:

$$\mathcal{H}(\delta P) = \begin{bmatrix} A^\top \delta PA - \delta P & A^\top \delta PB \\ B^\top \delta PA & B^\top \delta PB \end{bmatrix}$$

$$L(x,u) = \underbrace{l(x,u) - l(x_\mathrm{s}, u_\mathrm{s})}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top H \begin{bmatrix} x \\ u \end{bmatrix}} + \underbrace{\lambda(x) - \lambda(f(x,u))}_{\begin{bmatrix} x \\ u \end{bmatrix}^\top \mathcal{H}(\delta P) \begin{bmatrix} x \\ u \end{bmatrix}}$$

Solve the following **convex** SDP

$$\begin{aligned} \min_{\delta P, \alpha, \beta} \quad & \gamma\beta - \alpha \\ \text{s.t.} \quad & H + \mathcal{H}(\delta P) \succeq \alpha I, \\ & H + \mathcal{H}(\delta P) \preceq \beta I, \\ & \alpha \geq \epsilon. \end{aligned}$$

Solve

$$\begin{aligned} \min_{\delta P, F, \alpha, \beta} \quad & \gamma\beta - \alpha + \rho\|F\| \\ \text{s.t.} \quad & H + \mathcal{H}(\delta P) + C_{\mathbb{A}_\mathrm{s}}^\top F C_{\mathbb{A}_\mathrm{s}} \succeq \alpha I, \\ & \beta I \succeq H + \mathcal{H}(\delta P) + C_{\mathbb{A}_\mathrm{s}}^\top F C_{\mathbb{A}_\mathrm{s}}, \end{aligned}$$

ELMPC stabilising ⇒ ∃ SDP solution

PD LMPC ↔ PD tracking NMPC

PD LMPC ↔ PD tracking NMPC

We obtained

$$\text{ENMPC} \leftrightarrow \text{ELMPC} \leftrightarrow \text{PD LMPC}$$

PD LMPC $\leftrightarrow$ PD tracking NMPC

We obtained, but we also want

ENMPC $\leftrightarrow$ ELMPC $\leftrightarrow$ PD LMPC $\leftrightarrow$ PD tracking NMPC

PD LMPC $\leftrightarrow$ PD tracking NMPC

We obtained, but we also want

$$\text{ENMPC} \leftrightarrow \text{ELMPC} \leftrightarrow \text{PD LMPC} \leftrightarrow \text{PD tracking NMPC}$$

Lagrangian Hessian of the MPC Problems:

$$\nabla^2_{w_k}\mathcal{L}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s})$$

PD LMPC ↔ PD tracking NMPC

We obtained, but we also want

ENMPC ↔ ELMPC ↔ PD LMPC ↔ PD tracking NMPC

Lagrangian Hessian of the MPC Problems:

$$\nabla^2_{w_k}\mathcal{L}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s})$$

$$H := \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s})$$

PD LMPC ↔ PD tracking NMPC

We obtained, but we also want

ENMPC ↔ ELMPC ↔ PD LMPC ↔ PD tracking NMPC

Lagrangian Hessian of the MPC Problems:

$$\nabla^2_{w_k}\mathcal{L}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s})$$

$$\tilde{H} := \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) + \mathcal{H}(\delta P)$$

PD LMPC ↔ PD tracking NMPC

We obtained, but we also want

ENMPC ↔ ELMPC ↔ PD LMPC ↔ PD tracking NMPC

Lagrangian Hessian of the MPC Problems:

$$\nabla^2_{w_k}\mathcal{L}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s})$$

$$\tilde{H} := \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) + \mathcal{H}(\delta P)$$

$$\nabla^2_{w_k}\mathcal{L}^\mathrm{t}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \mathbf{2}\sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \mathbf{2}\sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) + \mathcal{H}(\delta P)$$

PD LMPC ↔ PD tracking NMPC

We obtained, but we also want

ENMPC ↔ ELMPC ↔ PD LMPC ↔ PD tracking NMPC

Lagrangian Hessian of the MPC Problems:

$$\nabla^2_{w_k}\mathcal{L}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s})$$

$$\tilde{H} := \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) + \mathcal{H}(\delta P)$$

$$\nabla^2_{w_k}\mathcal{L}^\mathrm{t}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \mathbf{2}\sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \mathbf{2}\sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) + \mathcal{H}(\delta P)$$

We know how to get $\tilde{H} \succ 0$

PD LMPC ↔ PD tracking NMPC

We obtained, but we also want

ENMPC ↔ ELMPC ↔ PD LMPC ↔ PD tracking NMPC

Lagrangian Hessian of the MPC Problems:

$$\nabla^2_{w_k}\mathcal{L}_k(w_s,\lambda_s,\mu_s) = \nabla^2 l(x_s,u_s) + \sum_{i=0}^{n_x}\lambda_{s,i}\nabla^2 f_i(x_s,u_s) - \sum_{j=0}^{n_\mu}\mu_{s,j}\nabla^2 h_j(x_s,u_s)$$

$$\tilde{H} := \nabla^2 l(x_s,u_s) + \sum_{i=0}^{n_x}\lambda_{s,i}\nabla^2 f_i(x_s,u_s) - \sum_{j=0}^{n_\mu}\mu_{s,j}\nabla^2 h_j(x_s,u_s) + \mathcal{H}(\delta P)$$

$$\nabla^2_{w_k}\mathcal{L}^{\mathrm{t}}_k(w_s,\lambda_s,\mu_s) = \nabla^2 l(x_s,u_s) + \mathbf{2}\sum_{i=0}^{n_x}\lambda_{s,i}\nabla^2 f_i(x_s,u_s) - \mathbf{2}\sum_{j=0}^{n_\mu}\mu_{s,j}\nabla^2 h_j(x_s,u_s) + \mathcal{H}(\delta P)$$

We know how to get $\tilde{H} \succ 0$, but in general

$$\tilde{H} - \sum_{i=0}^{n_x}\lambda_{s,i}\nabla^2 f_i(x_s,u_s) + \sum_{j=0}^{n_\mu}\mu_{s,j}\nabla^2 h_j(x_s,u_s) \nsucc 0$$

PD LMPC ↔ PD tracking NMPC

We obtained, but we also want

ENMPC ↔ ELMPC ↔ PD LMPC ↔ PD tracking NMPC

Lagrangian Hessian of the MPC Problems:

$$\nabla^2_{w_k}\mathcal{L}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s})$$

$$\tilde{H} := \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) + \mathcal{H}(\delta P)$$

$$\nabla^2_{w_k}\mathcal{L}^\mathrm{t}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \mathbf{2}\sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \mathbf{2}\sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) + \mathcal{H}(\delta P)$$

We know how to get $\tilde{H} \succ 0$, but in general

$$\tilde{H} - \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) + \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) \not\succ 0$$

Can rotate linearly with λ_s,

PD LMPC ↔ PD tracking NMPC

We obtained, but we also want

ENMPC ↔ ELMPC ↔ PD LMPC ↔ PD tracking NMPC

Lagrangian Hessian of the MPC Problems:

$$\nabla^2_{w_k}\mathcal{L}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s})$$

$$\tilde{H} := \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) + \mathcal{H}(\delta P)$$

$$\nabla^2_{w_k}\mathcal{L}^\mathrm{t}_k(w_\mathrm{s},\lambda_\mathrm{s},\mu_\mathrm{s}) = \nabla^2 l(x_\mathrm{s},u_\mathrm{s}) + \mathbf{2}\sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) - \mathbf{2}\sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) + \mathcal{H}(\delta P)$$

We know how to get $\tilde{H} \succ 0$, but in general

$$\tilde{H} - \sum_{i=0}^{n_x}\lambda_{\mathrm{s},i}\nabla^2 f_i(x_\mathrm{s},u_\mathrm{s}) + \sum_{j=0}^{n_\mu}\mu_{\mathrm{s},j}\nabla^2 h_j(x_\mathrm{s},u_\mathrm{s}) \not\succ 0$$

Can rotate linearly with λ_s, **can not** rotate with μ_s

PD LMPC $\leftrightarrow$ PD tracking NMPC

$$h(x, u) \geq 0 \qquad \Leftrightarrow \qquad h(x, u) - s = 0, \quad s \geq 0$$

PD LMPC ↔ PD tracking NMPC

$$h(x,u) \geq 0 \qquad \Leftrightarrow \qquad h(x,u) - s = 0, \quad s \geq 0$$

- s handled as ficticious controls

PD LMPC ↔ PD tracking NMPC

$$h(x, u) \geq 0 \qquad \Leftrightarrow \qquad h(x, u) - s = 0, \quad s \geq 0$$

- s handled as ficticious controls
- the inequalities do not contribute to the Hessian

PD LMPC $\leftrightarrow$ PD tracking NMPC

$$h(x,u) \geq 0 \qquad \Leftrightarrow \qquad h(x,u) - s = 0, \quad s \geq 0$$

- s handled as ficticious controls
- the inequalities do not contribute to the Hessian
- we can rotate wrt $h(x,u) - s$ beacuse it's an equality constraint

PD LMPC $\leftrightarrow$ PD tracking NMPC

$$h(x,u) \geq 0 \qquad \Leftrightarrow \qquad h(x,u) - s = 0, \quad s \geq 0$$

- s handled as ficticious controls
- the inequalities do not contribute to the Hessian
- we can rotate wrt $h(x,u) - s$ beacuse it's an equality constraint

We always have a gradient, so that the tracking stage cost must have a linear term too!

Evaporation Process

Evaporation Process

Model and cost

$$\begin{bmatrix} \dot{X}_2 \\ \dot{P}_2 \end{bmatrix} = f\left(\begin{bmatrix} X_2 \\ P_2 \end{bmatrix}, \begin{bmatrix} P_{100} \\ F_{200} \end{bmatrix} \right), \qquad \ell(x, u) = \text{something complicated.}$$

Evaporation Process

Model and cost

$$\begin{bmatrix} \dot{X}_2 \\ \dot{P}_2 \end{bmatrix} = f\left(\begin{bmatrix} X_2 \\ P_2 \end{bmatrix}, \begin{bmatrix} P_{100} \\ F_{200} \end{bmatrix} \right), \qquad \ell(x,u) = \text{something complicated.}$$

Bounds

$$X_2 \geq 25\,\%, \qquad 40\,\mathrm{kPa} \leq P_2 \leq 80\,\mathrm{kPa},$$
$$P_{100} \leq 400\,\mathrm{kPa}, \qquad F_{200} \leq 400 \mathrm{kg/min}.$$

Evaporation Process

Model and cost

$$\begin{bmatrix} \dot{X}_2 \\ \dot{P}_2 \end{bmatrix} = f\left(\begin{bmatrix} X_2 \\ P_2 \end{bmatrix}, \begin{bmatrix} P_{100} \\ F_{200} \end{bmatrix} \right), \qquad \ell(x,u) = \text{something complicated.}$$

Bounds

$$X_2 \geq 25\,\%, \qquad 40\,\mathrm{kPa} \leq P_2 \leq 80\,\mathrm{kPa},$$
$$P_{100} \leq 400\,\mathrm{kPa}, \qquad F_{200} \leq 400\mathrm{kg/min}.$$

Optimal steady state

$$\begin{bmatrix} X_2 \\ P_2 \end{bmatrix} = \begin{bmatrix} 25\,\% \\ 49.743\,\mathrm{kPa} \end{bmatrix}, \qquad \begin{bmatrix} P_{100} \\ F_{200} \end{bmatrix} = \begin{bmatrix} 191.713\,\mathrm{kPa} \\ 215.888\,\mathrm{kg/min} \end{bmatrix}.$$

Evaporation Process

Model and cost

$$\begin{bmatrix} \dot{X}_2 \\ \dot{P}_2 \end{bmatrix} = f\left(\begin{bmatrix} X_2 \\ P_2 \end{bmatrix}, \begin{bmatrix} P_{100} \\ F_{200} \end{bmatrix} \right), \qquad \ell(x, u) = \text{something complicated.}$$

Bounds

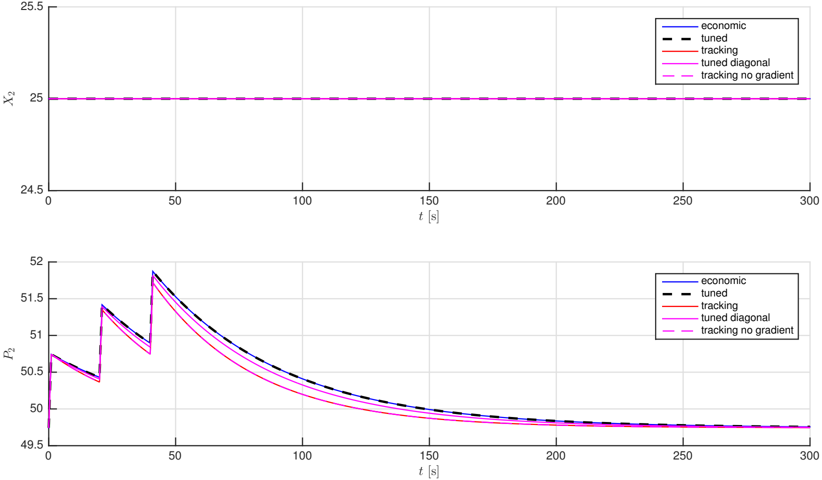
$$X_2 \geq 25\,\%, \qquad 40\,\mathrm{kPa} \leq P_2 \leq 80\,\mathrm{kPa},$$
$$P_{100} \leq 400\,\mathrm{kPa}, \qquad F_{200} \leq 400 \mathrm{kg/min}.$$

Optimal steady state

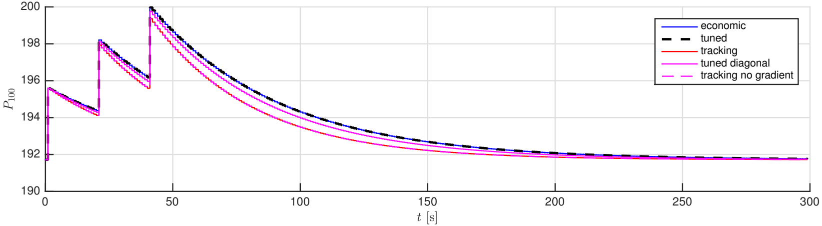
$$\begin{bmatrix} X_2 \\ P_2 \end{bmatrix} = \begin{bmatrix} 25\,\% \\ 49.743\,\mathrm{kPa} \end{bmatrix}, \qquad \begin{bmatrix} P_{100} \\ F_{200} \end{bmatrix} = \begin{bmatrix} 191.713\,\mathrm{kPa} \\ 215.888\,\mathrm{kg/min} \end{bmatrix}.$$

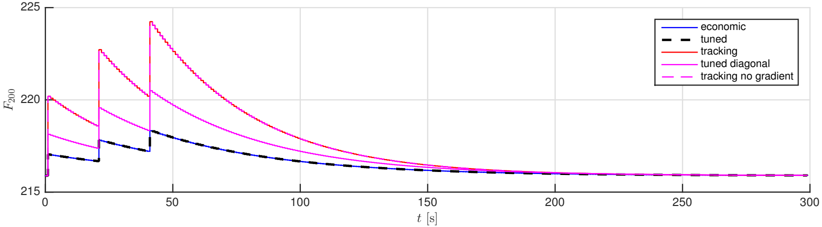
Compare: tuned (t), diagonal of tuned (td), normal (n), normal without gradient (n,ng)

Evaporation Process

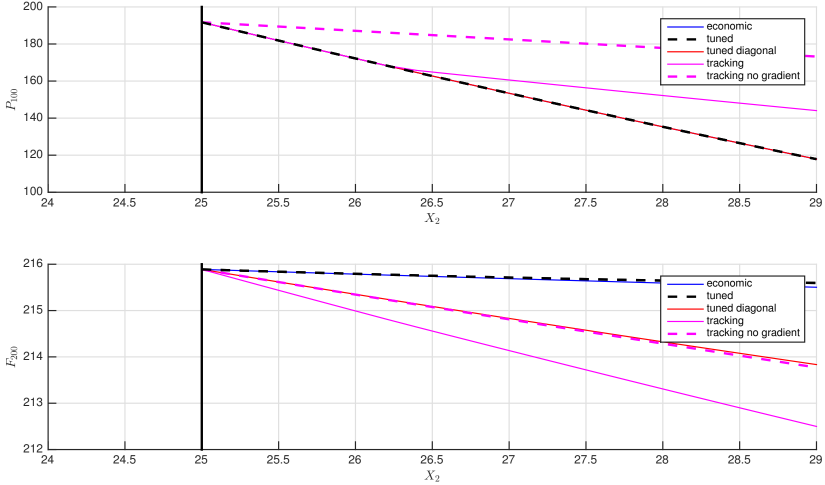


Evaporation Process

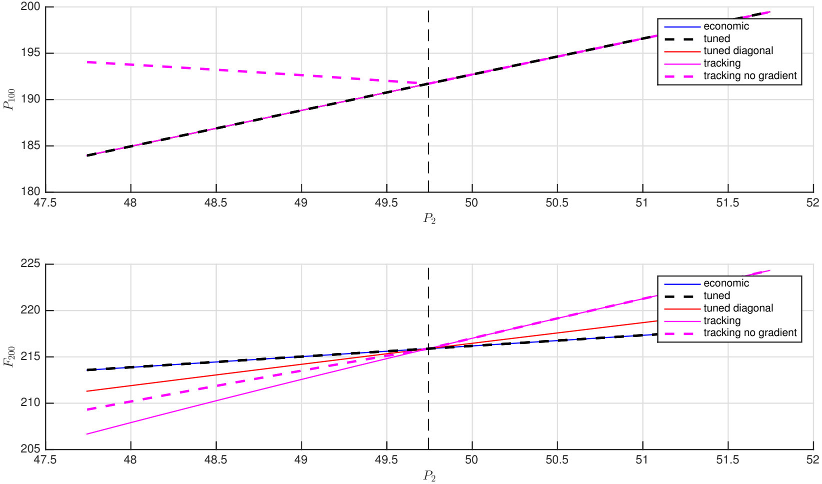




Evaporation Process



Evaporation Process



Evaporation Process

Optimality measure

$$G = \frac{\sum_{k=0}^{T-1} \ell_{\mathrm{eco}} - \sum_{k=0}^{T-1} \ell_{\mathrm{track}}}{T \ell_{\mathrm{ss}}}$$

Evaporation Process

Optimality measure

$$G = \frac{\sum_{k=0}^{T-1} \ell_{\mathrm{eco}} - \sum_{k=0}^{T-1} \ell_{\mathrm{track}}}{T \ell_{\mathrm{ss}}}$$

In our Example

$$G_{\mathrm{n}} = -3.2 \cdot 10^{-4}, \qquad G_{\mathrm{n,ng}} = -1.5 \cdot 10^{-3},$$

$$G_{\mathrm{td}} = -1.1 \cdot 10^{-4}, \qquad G_{\mathrm{t}} = -1.2 \cdot 10^{-7}.$$

Evaporation Process

Optimality measure

$$G = \frac{\sum_{k=0}^{T-1} \ell_{\text{eco}} - \sum_{k=0}^{T-1} \ell_{\text{track}}}{T \ell_{\text{ss}}}$$

In our Example

$$G_{\text{n}} = -3.2 \cdot 10^{-4}, \qquad G_{\text{n,ng}} = -1.5 \cdot 10^{-3},$$
$$G_{\text{td}} = -1.1 \cdot 10^{-4}, \qquad G_{\text{t}} = -1.2 \cdot 10^{-7}.$$

Profit loss over 1 year:

$$P_{\text{n,ng}} = -0.97 \text{ M\$/year}, \qquad P_{\text{n}} = -0.21 \text{ M\$/year},$$
$$P_{\text{td}} = -0.07 \text{ M\$/year}, \qquad P_{\text{t}} = -78 \text{ \$/year}.$$

Summary

- Economic MPC
  - Hard to solve numerically
  - Hard to prove stability
  - Best economic performance

Summary

- Economic MPC
  - Hard to solve numerically
  - Hard to prove stability
  - Best economic performance
- Tracking MPC
  - Stability enforcing
  - Easy to solve
  - Poor economic performance

Summary

- Economic MPC
  - Hard to solve numerically
  - Hard to prove stability
  - Best economic performance
- Tracking MPC
  - Stability enforcing
  - Easy to solve
  - Poor economic performance
- Tuned tracking MPC
  - Stability enforcing
  - Easy to solve
  - Good economic performance
  - Minimal modification wrt tracking MPC