IDENTIFICATION, ANALYSIS AND CONTROL OF DYNAMICAL SYSTEMS

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SCHOOL FOR ADVANCED STUDIES LUCCA

COURSE CONTENTS

1. Systems analysis (stability, controllability, observability), and synthesis of feedback controllers and state estimators

2. Systems identification (=get dynamical models from data)

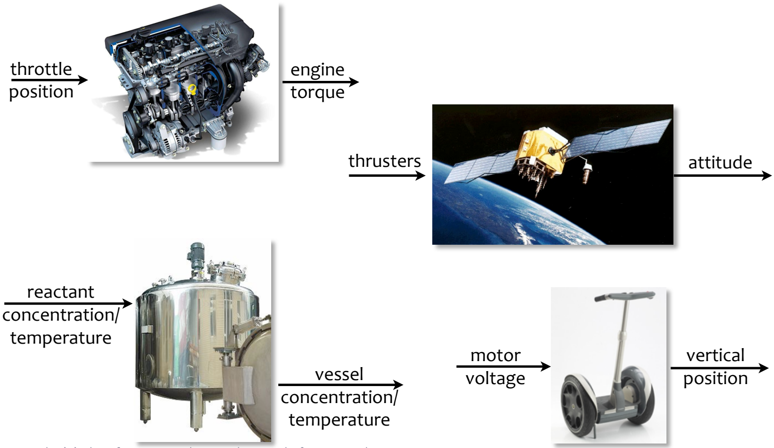
3. Analysis and control of linear parameter-varying systems

DYNAMICAL SYSTEMS

DYNAMICAL SYSTEMS

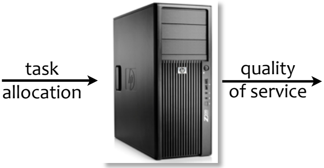
- A **dynamical system** is an object (or a set of objects) that evolves over time, possibly under external excitations.

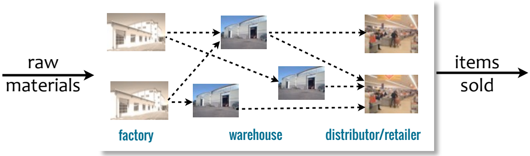
- Examples: an engine, a satellite, a tank reactor, a human transporter, ...



DYNAMICAL SYSTEMS

- ... a supply chain, a portfolio, a computer server





- The way the system evolves over time is called the **dynamics** of the system.

DYNAMICAL MODELS

- A **dynamical model** of a system is a set of mathematical laws that explain how the system evolves over time, usually under the effect of external excitations, in **quantitative** way.

- What is the purpose of a dynamical model ?

  1. Understand the system ("How does X influence Y ?")

  2. Simulation ("What happens if I apply action Z on the system ?")

  3. Estimate ("How to estimate variable X from measuring Y ?")

  4. Control ("How to make the system behave autonomously the way I want ?")

LINEAR SYSTEMS

CONTINUOUS-TIME LINEAR SYSTEMS

- System of n first-order differential equations with inputs

$$\left\{\begin{array}{rcll} \dot{x}_1(t) & = & a_{11}x_1(t) + \ldots + a_{1n}x_n(t) & +b_1u(t) \\ \dot{x}_2(t) & = & a_{21}x_1(t) + \ldots + a_{2n}x_n(t) & +b_2u(t) \\ \vdots & & \vdots & \vdots \\ \dot{x}_n(t) & = & a_{n1}x_1(t) + \ldots + a_{nn}x_n(t) & +b_nu(t) \\ x_1(0) = x_{10}, & \ldots & x_n(0) = x_{n0} & \end{array}\right.$$

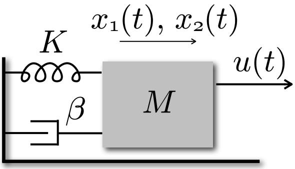
- Setting $x = [x_1 \ \ldots \ x_n]' \in \mathbb{R}^n$, the equivalent matrix form is the so-called **linear system**

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with initial condition

$$x(0) = x_0 = [x_{10} \ \ldots \ x_{n0}]' \in \mathbb{R}^n$$

EXAMPLE: MASS-SPRING-DAMPER SYSTEM



$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ M\dot{x}_2(t) = u - \beta x_2(t) - K x_1(t) \end{cases}$$

velocity = derivative of traveled space

Newton's law

Rewrite as the 2^{nd} order linear system

$$\begin{cases} \frac{dx_1(t)}{dt} = x_2(t) \\ \frac{dx_2(t)}{dt} = -\frac{\beta}{M}x_2(t) - \frac{K}{M}x_1(t) + \frac{1}{M}u(t) \end{cases}$$

or in matrix form

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{\beta}{M} \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{B} u(t)$$

n^{th}-ORDER LINEAR ODE WITH INPUT

$$\frac{dy^{(n)}(t)}{dt^n} + a_{n-1}\frac{dy^{(n-1)}(t)}{dt^{n-1}} + \cdots + a_1\dot{y}(t) + a_0 y(t)$$
$$= b_{n-1}\frac{du^{(n-1)}(t)}{dt} + b_{n-2}\frac{du^{(n-2)}(t)}{dt} + \cdots + b_1\dot{u}(t) + b_0 u(t)$$

By inspection the n^{th}-order ODE = 1$^{\text{st}}$-order linear system of ODEs

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \quad\vdots & \quad\vdots \\ \dot{x}_n(t) &= -a_0x_1(t) + \ldots - a_{n-1}x_n(t) + u(t) \\ y(t) &= b_0x_1(t) + \ldots + b_{n-1}x_n(t) \end{cases}$$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & b_2 & \ldots & b_{n-1} \end{bmatrix}, \; D = 0$$

The linear system of 1$^{\text{st}}$-order ODEs is called the **state-space realization** of the n^{th}-order ODE. There are infinitely many realizations.

LAGRANGE'S FORMULA

- For the continuous-time linear system $\dot{x} = Ax + Bu$ with initial condition $x(0) = x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t)$

$$x(t) = \underbrace{e^{At}x_0}_{\text{natural response}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced response}}$$

- The **exponential matrix** is defined as

$$e^{At} \triangleq I + At + \frac{A^2t^2}{2} + \ldots + \frac{A^nt^n}{n!} + \ldots$$

MATLAB
E=expm(A*t)

STATE VECTOR

- Given $x(0)$ and $u(t), \forall t \in [0,T]$, Lagrange's formula allows us to compute $x(t)$ and $y(t), \forall t \in [0,T]$
- Generally speaking, the **state** of a dynamical system is a set of variables that completely summarizes the past history of the system. It allows us to predict its future motion
- Therefore, by knowing the initial state $x(0)$ we can neglect all past history $u(-t), x(-t), \forall t \geq 0$
- The dimension n of the state $x(t) \in \mathbb{R}^n$ is called the **order** of the system

EIGENVALUES AND EIGENVECTORS

- Let us recall some basic concepts of linear algebra:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad \text{square matrix of order } n, \ A \in \mathbb{R}^{n\times n}$$

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad \text{identity matrix of order } n$$

- **Characteristic equation** of A:

$$\det(\lambda I - A) = 0$$

- **Characteristic polynomial** of A:

$$P(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$$

EIGENVALUES AND EIGENVECTORS

- The **eigenvalues** of $A \in \mathbb{R}^{n\times n}$ are the roots λ_1, ..., λ_n of its characteristic polynomial

$$\det(\lambda_i I - A) = 0, \quad i = 1, 2, \ldots, n$$

- An **eigenvector** of A is any vector $v_i \in \mathbb{R}^n$ such that $Av_i = \lambda_i v_i$ for some $i = 1, 2, \ldots, n$.
- The **diagonalization** of A is $A = T\Lambda T^{-1}$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = T^{-1}AT,\ T = [v_1|v_2|\ldots|v_n]$$

(not all matrices A are diagonalizable, see Jordan normal form)

- **Algebraic multiplicity** of λ_i = number of coincident roots λ_i of $\det(\lambda I - A)$
- **Geometric multiplicity** of λ_i = number of linearly independent eigenvectors v_i such that $Av_i = \lambda_i v_i$.

EIGENVALUES AND MODES

- Let $u(t) \equiv 0$ and assume A diagonalizable
- The state trajectory is the natural response

$$\begin{aligned} x(t) &= e^{At}x(0) = Te^{\Lambda t}\underbrace{T^{-1}x_0}_{\alpha} = [v_1 \ldots v_n]\begin{bmatrix} e^{\lambda_1 t} & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & e^{\lambda_n t} \end{bmatrix}\alpha \\ &= \begin{bmatrix} v_1 e^{\lambda_1 t} & \ldots & v_n e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i \end{aligned}$$

  where v_i=eigenvector of A, λ_i=eigenvalue of A, $\alpha = T^{-1}x(0) \in \mathbb{R}^n$

- The evolution of the system depends on the eigenvalues λ_i of A, called **modes** of the system (sometimes we also refer to $e^{\lambda_i t}$ as the i-th mode)
- A mode λ_i is called **excited** if $\alpha_i \neq 0$

SOME CLASSES OF DYNAMICAL SYSTEMS

- **Causality**: a dynamical system is **causal** if $y(t)$ does not depend on future inputs $u(\tau)\ \forall\tau > t$ (**strictly causal** if $\forall\tau \geq t$)
- A linear system is always causal, and strictly causal iff $D = 0$
- **Linear time-varying (LTV) systems**:

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + B(t)u(t) \\ y(t) & = & C(t)x(t) + D(t)u(t) \end{array}\right.$$

- When A, B, C, D are constant, the system is said **linear time-invariant** (LTI)
- **Multivariable systems**: more generally, a system can have m inputs ($u(t) \in \mathbb{R}^m$) and p outputs ($y(t) \in \mathbb{R}^p$). For linear systems, we still have

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array}\right.$$

with

$$A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$$

SOME CLASSES OF DYNAMICAL SYSTEMS

- **Nonlinear systems**

$$\begin{cases} \dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & g(x(t), u(t)) \end{cases}$$

where $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $g : \mathbb{R}^{n+m} \to \mathbb{R}^p$ are (arbitrary) nonlinear functions

- **Time-varying nonlinear systems** are very general classes of dynamical systems

$$\begin{cases} \dot{x}(t) & = & f(t, x(t), u(t)) \\ y(t) & = & g(t, x(t), u(t)) \end{cases}$$

STABILITY

EQUILIBRIUM

- Consider the continuous-time nonlinear system

$$\begin{cases} \dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & g(x(t), u(t)) \end{cases}$$

Definition

A state $x_r \in \mathbb{R}^n$ and an input $u_r \in \mathbb{R}^m$ are an **equilibrium pair** if for initial condition $x(0) = x_r$ and constant input $u(t) \equiv u_r$ the state remains constant: $x(t) \equiv x_r, \forall t \geq 0$.

- Equivalent definition: (x_r, u_r) is an equilibrium pair if $f(x_r, u_r) = 0$
- x_r is called **equilibrium state**, u_r **equilibrium input**
- The definition generalizes to time-varying nonlinear systems

STABILITY

- Consider the nonlinear system

$$\begin{cases} \dot{x}(t) &= f(x(t), u_r) \\ y(t) &= g(x(t), u_r) \end{cases}$$

and let x_r an equilibrium state, $f(x_r, u_r) = 0$

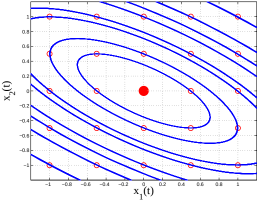
Definition

The equilibrium state x_r is **stable** if for each initial conditions $x(0)$ "close enough" to x_r, the corresponding trajectory $x(t)$ remains near x_r for all $t \geq 0$. [a]

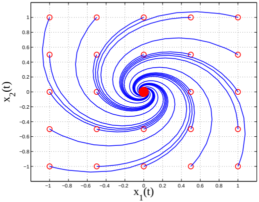
[a] Analytic definition: $\forall \epsilon > 0\, \exists \delta > 0: \|x(0) - x_r\| < \delta \Rightarrow \|x(t) - x_r\| < \epsilon, \forall t \geq 0$.

- The equilibrium point x_r is called **asymptotically stable** if it is stable and $x(t) \to x_r$ for $t \to \infty$
- Otherwise, the equilibrium point x_r is called **unstable**

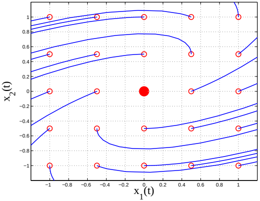
STABILITY OF EQUILIBRIA - EXAMPLES



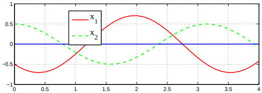
stable equilibrium

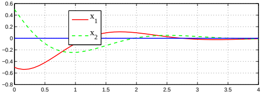


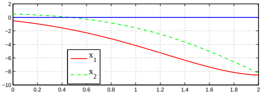
asymptotically stable equilibrium



unstable equilibrium







$$\frac{dx}{dt} = \begin{bmatrix} -2x_1(t) - 4x_2(t) \\ 2x_1(t) + 2x_2(t) \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{bmatrix} -x_1(t) - 2x_2(t) \\ 2x_1(t) - x_2(t) \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{bmatrix} 2x_1(t) - 2x_2(t) \\ x_1(t) \end{bmatrix}$$

STABILITY OF FIRST-ORDER LINEAR SYSTEMS

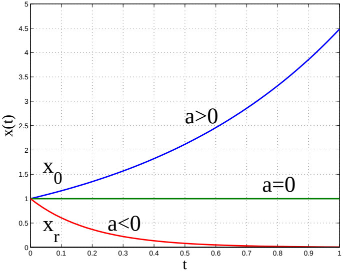
- Consider the first-order linear system

$$\dot{x}(t) = ax(t) + bu(t)$$

- $x_r = 0, u_r = 0$ is an equilibrium pair
- For $u(t) \equiv 0, \forall t \geq 0$, the solution is

$$x(t) = e^{at} x_0$$

- The origin $x_r = 0$ is
  - unstable if $a > 0$
  - stable if $a \leq 0$
  - asymptotically stable if $a < 0$



STABILITY OF CONTINUOUS-TIME LINEAR SYSTEMS

Since the natural response of $\dot{x} = Ax + Bu$ is $x(t) = e^{At}x_0$, the stability properties depend only on A. We can therefore talk about **system stability** of a linear system (A, B, C, D)

Theorem

Let $\lambda_1, \ldots, \lambda_m$, $m \leq n$ be the eigenvalues of $A \in \mathbb{R}^{n\times n}$. The system $\dot{x} = Ax + Bu$ is

- asymptotically stable iff $\Re\lambda_i < 0, \forall i = 1, \ldots, m$
- (marginally) stable if $\Re\lambda_i \leq 0, \forall i = 1, \ldots, m$, and the eigenvalues with null real part have equal algebraic and geometric multiplicity
- unstable if $\exists\, i$ such that $\Re\lambda_i > 0$.

The stability properties of a linear system only depend on the **real part** of the eigenvalues of matrix A

Proof:

- The natural response is $x(t)=e^{At}x_0$ ($e^{At} \triangleq I+At+\frac{A^2t^2}{2}+\ldots+\frac{A^nt^n}{n!}+\ldots$)
- If matrix A is diagonalizable[1] , $A=T\Lambda T^{-1}$,

$$\Lambda=\begin{bmatrix}\lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda_n\end{bmatrix} \Rightarrow e^{At}=T\begin{bmatrix}e^{\lambda_1 t} & 0 & \ldots & 0\\ 0 & e^{\lambda_2 t} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & e^{\lambda_n t}\end{bmatrix}T^{-1}$$

- Take any eigenvalue $\lambda=a+jb$:

$$|e^{\lambda t}|=e^{at}|e^{jbt}|=e^{at}$$

- A is always diagonalizable if algebraic multiplicity = geometric multiplicity

$\square$

[1] If A is not diagonalizable, it can be transformed to Jordan form. In this case the natural response $x(t)$ contains modes $t^j e^{\lambda t}$, $j=0,1,\ldots,$ alg. multiplicity - geom. multiplicity

LINEARIZATION OF NONLINEAR SYSTEMS

- Consider the nonlinear system

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & g(x(t), u(t))\end{array}\right.$$

- Let (x_r, u_r) be an equilibrium, $f(x_r, u_r) = 0$
- Objective: investigate the dynamic behaviour of the system for small perturbations $\Delta u(t) \triangleq u(t) - u_r$ and $\Delta x(0) \triangleq x(0) - x_r$.
- The evolution of $\Delta x(t) \triangleq x(t) - x_r$ is given by

$$\begin{aligned}\dot{\Delta} x(t) &= \dot{x}(t) - \dot{x}_r = f(x(t), u(t)) \\ &= f(\Delta x(t) + x_r, \Delta u(t) + u_r) \\ &\approx \underbrace{\frac{\partial f}{\partial x}(x_r, u_r)}_{A} \Delta x(t) + \underbrace{\frac{\partial f}{\partial u}(x_r, u_r)}_{B} \Delta u(t)\end{aligned}$$

- Similarly

$$\Delta y(t) \approx \underbrace{\frac{\partial g}{\partial x}(x_r, u_r)}_{C} \Delta x(t) + \underbrace{\frac{\partial g}{\partial u}(x_r, u_r)}_{D} \Delta u(t)$$

  where $\Delta y(t) \triangleq y(t) - g(x_r, u_r)$ is the perturbation of the output from its equilibrium

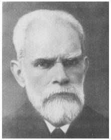
- The perturbations $\Delta x(t)$, $\Delta y(t)$, and $\Delta u(t)$ are (approximately) ruled by the **linearized system**

$$\begin{cases} \dot{\Delta} x(t) &= A\Delta x(t) + B\Delta u(t) \\ \Delta y(t) &= C\Delta x(t) + D\Delta u(t) \end{cases}$$

LYAPUNOV'S STABILITY

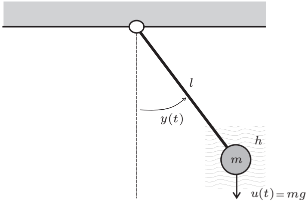
LYAPUNOV'S INDIRECT METHOD

- Consider the nonlinear system $\dot{x} = f(x)$, with f differentiable, and assume $x = 0$ is equilibrium point ($f(0) = 0$)
- Consider the linearized system $\dot{x} = Ax$, with $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$
- If $\dot{x} = Ax$ is asymptotically stable, then the origin $x = 0$ is also an asymptotically stable equilibrium for the nonlinear system (locally)
- If $\dot{x} = Ax$ is unstable, then the origin $x = 0$ is an unstable equilibrium for the nonlinear system
- If A is marginally stable, nothing can be said about the stability of the origin $x = 0$ for the nonlinear system



Aleksandr Mikhailovich Lyapunov
(1857-1918)

EXAMPLE: PENDULUM



$y(t)$ = angular displacement
$\dot{y}(t)$ = angular velocity
$\ddot{y}(t)$ = angular acceleration
$u(t)$ = mg gravity force
$h\dot{y}(t)$ = viscous friction torque
l = pendulum length
ml^2 = pendulum rotational inertia

- mathematical model

$$ml^2\ddot{y}(t) = -lmg\sin y(t) - h\dot{y}(t)$$

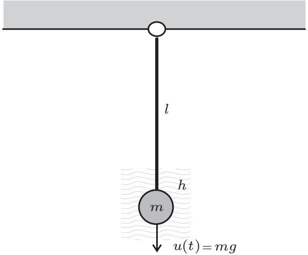
- in state-space form ($x_1 = y, x_2 = \dot{y}$)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 - Hx_2, \quad H \triangleq \frac{h}{ml^2} \end{cases}$$

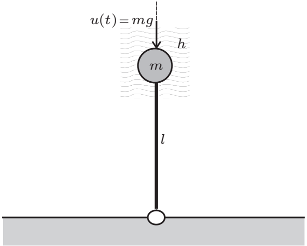
EXAMPLE: PENDULUM

Look for equilibrium states:

$$\begin{bmatrix} x_{2r} \\ -\frac{g}{l}\sin x_{1r} - Hx_{2r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_{2r} = 0 \\ x_{1r} = \pm k\pi,\ k = 0, 1, \ldots \end{cases}$$



$$x_{2r} = 0, x_{1r} = 0, \pm 2\pi, \ldots$$



$$x_{2r} = 0, x_{1r} = 0, \pm\pi, \pm 3\pi, \ldots$$

EXAMPLE: PENDULUM

- Linearize the system around $x_{1r} = 0, x_{2r} = 0$

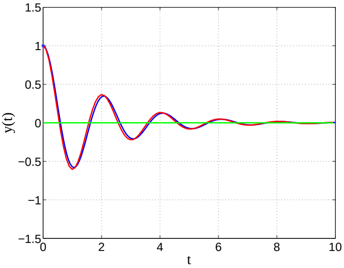
$$\Delta\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -H \end{bmatrix}}_{A} \Delta x(t)$$

- find the eigenvalues of A

$$\det(\lambda I - A) = \lambda^2 + H\lambda + \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{1}{2}\left(-H \pm \sqrt{H^2 - 4\frac{g}{l}}\right)$$

- $\Re\lambda_{1,2} < 0 \Rightarrow \dot{x} = Ax$ asymptotically stable

- by Lyapunov's indirect method $x_r = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is also an asymptotically stable equilibrium for the pendulum



EXAMPLE: PENDULUM

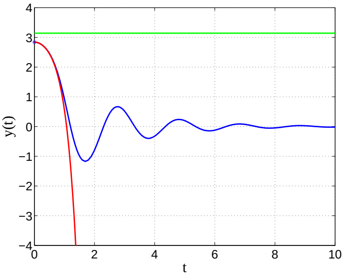
- Linearize the system around $x_{1r} = \pi, x_{2r} = 0$

$$\Delta\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -H \end{bmatrix}}_{A} \Delta x(t)$$

- find the eigenvalues of A

$$\det(\lambda I - A) = \lambda^2 + H\lambda - \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{1}{2}\left(-H \pm \sqrt{H^2 + 4\frac{g}{l}}\right)$$

- $\lambda_1 < 0, \lambda_2 > 0 \Rightarrow \dot{x} = Ax$ unstable
- by Lyapunov's indirect method $x_r = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is also an unstable equilibrium for the pendulum



LYAPUNOV'S DIRECT METHOD

- A second method exists to analyze global stability of nonlinear systems, based on the concept of **Lyapunov functions**
- **Key idea**: if the energy of a system dissipates over time, the system asymptotically reaches a minimum-energy configuration
- Assumptions: consider the autonomous nonlinear system $\dot{x} = f(x)$, with $f(\cdot)$ differentiable, and let $x = 0$ be an equilibrium ($f(0) = 0$)
- Some definitions of positive definiteness of a function $V : \mathbb{R}^n \mapsto \mathbb{R}$
  - V is called **locally positive definite** if $V(0) = 0$ and there exists a **ball** $B_\epsilon = \{x : \|x\|_2 \leq \epsilon\}$ around the origin such that $V(x) > 0 \, \forall x \in B_\epsilon \setminus 0$
  - V is called **globally positive definite** if $B_\epsilon = \mathbb{R}^n$ (i.e. $\epsilon \to \infty$)
  - V is called **negative definite** if $-V$ is positive definite
  - V is called **positive semi-definite** if $V(x) \geq 0 \, \forall x \in B_\epsilon, x \neq 0$
  - V is called **positive semi-negative** if $-V$ is positive semi-definite

- Example: let $x = [x_1 \; x_2]', V : \mathbb{R}^2 \to \mathbb{R}$
  - $V(x) = x_1^2 + x_2^2$ is globally positive definite
  - $V(x) = x_1^2 + x_2^2 - x_1^3$ is locally positive definite
  - $V(x) = x_1^4 + \sin^2(x_2)$ is locally positive definite and globally positive semi-definite

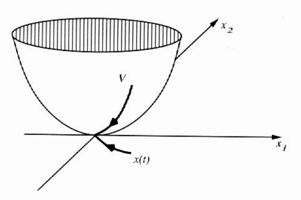
LYAPUNOV'S DIRECT METHOD

Theorem

Given the nonlinear system $\dot{x} = f(x)$, $f(0) = 0$, let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be positive definite in a ball B_ϵ around the origin, $\epsilon > 0$, $V \in C^1(\mathbb{R})$. If the function

$$\dot{V}(x) = \nabla V(x)'\dot{x} = \nabla V(x)' f(x)$$

is negative definite on B_ϵ, then the origin is an asymptotically stable equilibrium point with **domain of attraction** B_ϵ ($\lim_{t\to+\infty} x(t) = 0$ for all $x(0) \in B_\epsilon$). If $\dot{V}(x)$ is only negative semi-definite on B_ϵ, then the the origin is a stable equilibrium point.



Such a function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is called a **Lyapunov function** for the system $\dot{x} = f(x)$

EXAMPLE OF LYAPUNOV'S DIRECT METHOD

- Consider the following nonlinear system $\dot{x} = f(x)$ given by

$$\begin{cases} \dot{x}_1 & = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2 \\ \dot{x}_2 & = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2) \end{cases}$$

- The state $x = 0$ is an equilibrium because $\dot{x} = f(0) = 0$
- Consider the candidate Lyapunov function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

  which is globally positive definite. Its time derivative $\dot{V}$ is

$$\dot{V}(x_1, x_2) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2)$$

- It is easy to check that $\dot{V}(x_1, x_2)$ is negative definite if $\|x\|_2^2 = x_1^2 + x_2^2 < 2$
- Since for any B_ϵ with $0 < \epsilon < \sqrt{2}$ the hypotheses of Lyapunov's theorem are satisfied, $x = 0$ is an asymptotically stable equilibrium
- Any B_ϵ with $0 < \epsilon < \sqrt{2}$ is a domain of attraction

EXAMPLE OF LYAPUNOV'S DIRECT METHOD (CONT'D)

- Cf. Lyapunov's indirect method: the linearization around $x = 0$ is

$$\frac{\partial f(0,0)}{\partial x} = \left. \begin{bmatrix} 3x_1^2 - 3x_2^2 - 2 & -6x_1x_2 \\ 10x_1x_2 & 5x_1^2 + 3x_2^2 - 2 \end{bmatrix} \right|_{x=0} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

  which is an asymptotically stable matrix
- Lyapunov's indirect method tells us that the origin is <u>locally</u> asymptotically stable
- Lyapunov's direct method also tells us that B_ϵ is a domain of attraction for all $0 < \epsilon < \sqrt{2}$

- Consider this other example: $\dot{x} = -x^3$. The origin as an equilibrium. But $\frac{\partial f(0,0)}{\partial x} = -3 \cdot 0^2 = 0$, so Lyapunov indirect method is useless.
- Lyapunov's direct method with $V = x^2$ provides $\dot{V} = -2x^4$, and therefore we can conclude that $x = 0$ is (globally) asymptotically stable

CASE OF CONTINUOUS-TIME LINEAR SYSTEMS

- Let us apply Lyapunov's direct method to linear systems $\dot{x} = Ax$ and choose $V(x) = x'Px$, with $P = P' \succ 0$ (P=positive definite and symmetric matrix)
- The derivative $\dot{V}(x) = \dot{x}'Px + x'P\dot{x} = x'(A'P + PA)x$
- $\dot{V}(x)$ is negative definite if and only if the **Lyapunov equation**

$$A'P + PA = -Q$$

is satisfied for some $Q \succ 0$ (for example, $Q = I$)

Theorem

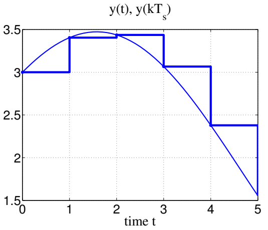
The autonomous linear system $\dot{x} = Ax$ is asymptotically stable $\Leftrightarrow \forall Q \succ 0$ the Lyapunov equation $A'P + PA = -Q$ has one and only one solution $P \succ 0$

MATLAB
»P=lyap(A',Q)

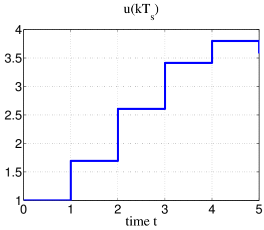
$\longleftarrow$ Note the transposition of matrix A !

DISCRETE-TIME SYSTEMS

DISCRETE-TIME MODELS



Sampling of a continuous signal



Discrete-time signal

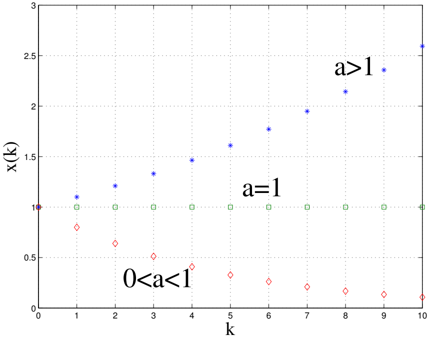
- Discrete-time models describe relationships between **sampled** variables $x(kT_s), u(kT_s), y(kT_s), k = 0, 1, \ldots$
- The value $u(kT_s)$ is kept constant during the **sampling interval** $[kT_s, (k+1)T_s)$
- A discrete-time signal can either represent the **sampling** of a **continuous-time** signal, or be an intrinsically discrete signal
- Discrete-time signals are at the basis of **digital controllers** (as well as of digital filters in signal processing)

DIFFERENCE EQUATION

- Consider the first order **difference equation** (autonomous system)

$$\begin{cases} x(k+1) &= ax(k) \\ x(0) &= x_0 \end{cases}$$

- The solution is $x(k) = a^k x_0$



LINEAR DISCRETE-TIME SYSTEM

- Consider the set of n first-order linear difference equations forced by the input $u(k) \in \mathbb{R}$

$$\begin{cases} x_1(k+1) &=& a_{11}x_1(k) + \ \ldots + a_{1n}x_n(k) & +b_1u(k) \\ x_2(k+1) &=& a_{21}x_1(k) + \ \ldots + a_{2n}x_n(k) & +b_2u(k) \\ \vdots & \vdots & & \vdots \\ x_n(k+1) &=& a_{n1}x_1(k) + \ \ldots + a_{nn}x_n(k) & +b_nu(k) \\ x_1(0) = x_{10}, & \ldots & x_n(0) = x_{n0} \end{cases}$$

- In compact matrix form:

$$\begin{cases} x(k+1) &=& Ax(k) + Bu(k) \\ x(0) &=& x_0 \end{cases}$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

LINEAR DISCRETE-TIME SYSTEM

- The solution is

$$x(k) = \underbrace{A^k x_0}_{\text{natural response}} + \underbrace{\sum_{i=0}^{k-1} A^i B u(k-1-i)}_{\text{forced response}}$$

- If matrix A is diagonalizable, $A = T\Lambda T^{-1}$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \Rightarrow A^k = T \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix} T^{-1}$$

where $T = [v_1 \dots v_n]$ collects n independent eigenvectors.

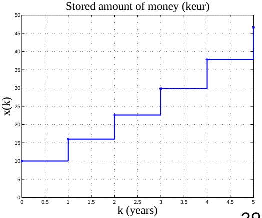
EXAMPLE - WEALTH OF A BANK ACCOUNT

- k= year counter
- ρ= interest rate
- $x(k)$= wealth at the beginning of year k
- $u(k)$= money saved at the end of year k
- x_0= initial wealth in bank account

Discrete-time model:

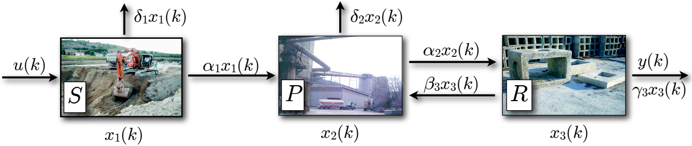
$$\begin{cases} x(k+1) = (1+\rho)x(k) + u(k) \\ x(0) = x_0 \end{cases}$$

x_0	10 k€
$u(k)$	5 k€
ρ	10 %



$$x(k) = (1.1)^k \cdot 10 + \frac{1-(1.1)^k}{1-1.1} 5 = 60(1.1)^k - 50$$

EXAMPLE - SUPPLY CHAIN



- Problem statement:
  - At each month k, S purchases the quantity $u(k)$ of raw material
  - A fraction δ_1 of raw material is discarded, a fraction α_1 is shipped to producer P
  - A fraction α_2 of product is sold by P to retailer R, a fraction δ_2 is discarded
  - Retailer R returns a fraction β_3 of defective products every month and sells a fraction γ_3 to customers

- Mathematical model:

$$\left\{\begin{array}{rcl} x_1(k+1) & = & (1-\alpha_1-\delta_1)x_1(k)+u(k) \\ x_2(k+1) & = & \alpha_1 x_1(k)+(1-\alpha_2-\delta_2)x_2(k) \\ & & +\beta_3 x_3(k) \\ x_3(k+1) & = & \alpha_2 x_2(k)+(1-\beta_3-\gamma_3)x_3(k) \\ y(k) & = & \gamma_3 x_3(k) \end{array}\right.$$

k	month counter
$x_1(k)$	raw material in S
$x_2(k)$	products in P
$x_3(k)$	products in R
$y(k)$	products sold to customers

EXAMPLE - STUDENT POPULATION DYNAMICS

- Problem statement:
  - 3-years course
  - percentage of promoted, repeaters, and dropouts are roughly constant
  - direct enrollment in 2nd and 3rd academic year is not allowed
  - students cannot enroll for more than 3 years
- Notation:

k	Year
$x_i(k)$	Number of students enrolled in year i at year k, $i = 1, 2, 3$
$u(k)$	Number of freshmen at year k
$y(k)$	Number of graduates at year k
α_i	promotion rate during year i, $0 \leq \alpha_i \leq 1$
β_i	failure rate during year i, $0 \leq \beta_i \leq 1$
γ_i	dropout rate during year i, $\gamma_i = 1 - \alpha_i - \beta_i \geq 0$

- 3^{rd}-order linear discrete-time system:

$$\left\{\begin{array}{rcl} x_1(k+1) & = & \beta_1 x_1(k) + u(k) \\ x_2(k+1) & = & \alpha_1 x_1(k) + \beta_2 x_2(k) \\ x_3(k+1) & = & \alpha_2 x_2(k) + \beta_3 x_3(k) \\ y(k) & = & \alpha_3 x_3(k) \end{array}\right.$$

EXAMPLE - STUDENT POPULATION DYNAMICS

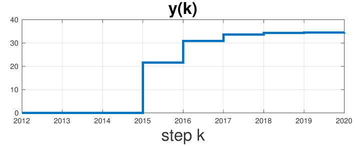
- In matrix form

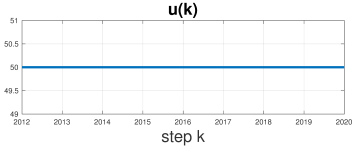
$$\begin{cases} x(k+1) &= \begin{bmatrix} \beta_1 & 0 & 0 \\ \alpha_1 & \beta_2 & 0 \\ 0 & \alpha_2 & \beta_3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & \alpha_3 \end{bmatrix} x(k) \end{cases}$$

- Simulation

$\alpha_1 = .60$	$\beta_1 = .20$
$\alpha_2 = .80$	$\beta_2 = .15$
$\alpha_3 = .90$	$\beta_3 = .08$

$$u(k) \equiv 50, k = 2012, \ldots$$





- Consider the n^{th}-**order difference equation** forced by u

$$
\begin{aligned}
a_n y(k-n) + a_{n-1} y(k-n+1) + \cdots + a_1 y(k-1) + y(k) \\
= b_n u(k-n) + \cdots + b_1 u(k-1) + b_0 u(k)
\end{aligned}
$$

- Equivalent linear discrete-time system in **canonical state matrix form**

$$
\left\{
\begin{aligned}
x(k+1) &= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} (b_n - b_0 a_n) & \ldots & (b_1 - b_0 a_1) \end{bmatrix} x(k) + b_0 u(k)
\end{aligned}
\right.
$$

- There are infinitely many state-space realizations

MATLAB
tf2ss

- n^{th}-order difference equations are very useful for digital filters, digital controllers, and to reconstruct models from data (**system identification**)

MODAL RESPONSE

- Assume input $u(k) = 0, \forall k \geq 0$
- Assume A is diagonalizable, $A = T\Lambda T^{-1}$
- The state trajectory (natural response) is

$$x(k) = A^k x_0 = T\Lambda^k T^{-1} x_0 = \sum_{i=1}^{n} \alpha_i \lambda_i^k v_i$$

  where

  - λ_i = eigenvalues of A
  - v_i = eigenvectors of A
  - α_i = coefficients that depend on the initial condition $x(0)$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = T^{-1}x(0),\ T = [v_1 \ldots v_n]$$

- The system modes depend on the eigenvalues of A, as in continuous-time

DISCRETE-TIME LINEAR SYSTEM

$$\left\{\begin{array}{rcl} x(k+1) & = & Ax(k)+Bu(k) \\ y(k) & = & Cx(k)+Du(k) \\ x(0) & = & x_0 \end{array}\right.$$

- From a given initial condition $x(0)$ and input sequence $\{u(k)\}_{k=0}^{\infty}$ one can predict the entire sequence of states $x(k)$ and outputs $y(k)$, $\forall k \in \mathbb{N}$
- The state $x(0)$ summarizes all the past history of the system
- The dimension n of the state $x(k) \in \mathbb{R}^n$ is called the **order** of the system
- The system is called **proper** (or **strictly causal**) if $D = 0$
- General multivariable case:

$$\begin{array}{rcl} x(k) & \in & \mathbb{R}^n \\ u(k) & \in & \mathbb{R}^m \\ y(k) & \in & \mathbb{R}^p \end{array} \qquad \begin{array}{rcl} A & \in & \mathbb{R}^{n\times n} \\ B & \in & \mathbb{R}^{n\times m} \\ C & \in & \mathbb{R}^{p\times n} \\ D & \in & \mathbb{R}^{p\times m} \end{array}$$

EQUILIBRIUM

- Consider the discrete-time nonlinear system

$$\left\{\begin{array}{rcl} x(k+1) & = & f(x(k),u(k)) \\ y(k) & = & g(x(k),u(k)) \end{array}\right.$$

Definition

A state $x_r \in \mathbb{R}^n$ and an input $u_r \in \mathbb{R}^m$ are an **equilibrium pair** if for initial condition $x(0) = x_r$ and constant input $u(k) \equiv u_r$, $\forall k \in \mathbb{N}$, the state remains constant: $x(k) \equiv x_r, \forall k \in \mathbb{N}$.

- Equivalent definition: (x_r, u_r) is an equilibrium pair if $f(x_r, u_r) = x_r$
- x_r is called **equilibrium state**, u_r **equilibrium input**
- The definition generalizes to time-varying discrete-time nonlinear systems

STABILITY

- Consider the nonlinear system

$$\begin{cases} x(k+1) &= f(x(k), u_r) \\ y(k) &= g(x(k), u_r) \end{cases}$$

and let x_r an equilibrium state, $f(x_r, u_r) = x_r$

Definition

The equilibrium state x_r is **stable** if for each initial conditions $x(0)$ "close enough" to x_r, the corresponding trajectory $x(k)$ remains near x_r for all $k \in \mathbb{N}$. [a]

[a] Analytic definition: $\forall \epsilon > 0 \, \exists \delta > 0 : \|x(0) - x_r\| < \delta \Rightarrow \|x(k) - x_r\| < \epsilon, \forall k \in \mathbb{N}$.

- The equilibrium point x_r is called **asymptotically stable** if it is stable and $x(k) \to x_r$ for $k \to \infty$
- Otherwise, the equilibrium point x_r is called **unstable**

STABILITY OF FIRST-ORDER LINEAR SYSTEMS

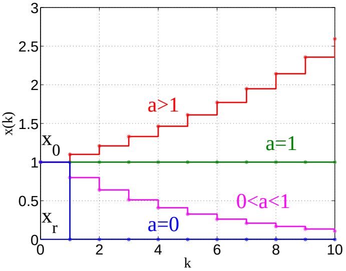
- Consider the first-order linear system

$$x(k+1) = ax(k) + bu(k)$$

- $x_r = 0, u_r = 0$ is an equilibrium pair
- For $u(k) \equiv 0, \forall k = 0, 1, \ldots$, the solution is

$$x(k) = a^k x_0$$

- The origin $x_r = 0$ is
  - unstable if $|a| > 1$
  - stable if $|a| \leq 1$
  - asymptotically stable if $|a| < 1$



STABILITY OF DISCRETE-TIME LINEAR SYSTEMS

The natural response of $x(k+1) = Ax(k) + Bu(k)$ is $x(k) = A^k x_0$, so stability only depend on A. We therefore talk about **system stability**

Theorem

Let $\lambda_1, \ldots, \lambda_m, m \leq n$ be the eigenvalues of $A \in \mathbb{R}^{n \times n}$. The system $x(k+1) = Ax(k) + Bu(k)$ is

- asymptotically stable iff $|\lambda_i| < 1, \forall i = 1, \ldots, m$
- (marginally) stable if $|\lambda_i| \leq 1, \forall i = 1, \ldots, m$, and the eigenvalues with unit modulus have equal algebraic and geometric multiplicity [a]
- unstable if $\exists\, i$ such that $|\lambda_i| > 1$

[a] Algebraic multiplicity of λ_i = number of coincident roots λ_i of $\det(\lambda I - A)$. Geometric multiplicity of λ_i = number of linearly independent eigenvectors v_i, $Av_i = \lambda_i v_i$

The stability properties of a discrete-time linear system only depend on the **modulus** of the eigenvalues of matrix A

Proof:

- The natural response is $x(k) = A^k x_0$
- If matrix A is diagonalizable[2] , $A = T\Lambda T^{-1}$,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \Rightarrow A^k = T \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix} T^{-1}$$

- Take any eigenvalue $\lambda = \rho e^{j\theta}$:

$$|\lambda^k| = \rho^k |e^{jk\theta}| = \rho^k$$

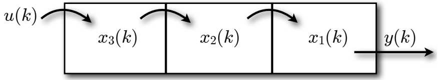
- A is always diagonalizable if algebraic multiplicity - geometric multiplicity

$\square$

[2] If A is not diagonalizable, it can be transformed to Jordan form. In this case the natural response $x(t)$ contains modes $k^j \lambda^k, j = 0, 1, \dots,$ alg. multiplicity = geom. multiplicity

ZERO EIGENVALUES

- Modes λ_i=0 determine finite-time convergence to zero.
- This has no continuous-time counterpart, where instead all converging modes tend to zero in infinite time ($e^{\lambda_i t}$)
- Example: dynamics of a buffer



$$\left\{\begin{array}{rcl} x_1(k+1) & = & x_2(k) \\ x_2(k+1) & = & x_3(k) \\ x_3(k+1) & = & u(k) \\ y(k) & = & x_1(k) \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} x(k+1) & = & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k) \\ y(k) & = & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k) \end{array}\right.$$

- Natural response: $A^3 x(0) = 0$ for all $x(0) \in \mathbb{R}^3$
- For $u(k) \equiv 0$, the buffer deploys after at most 3 steps !

EXACT SAMPLING

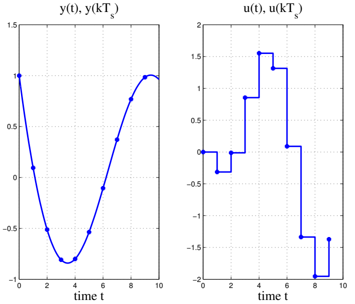
- Consider the continuos-time system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ x(0) &= x_0 \end{cases}$$

- We want to characterize the value of $x(t), y(t)$ at the time instants $t = 0, T_s, 2T_s, \ldots, kT_s, \ldots$, **under the assumption that the input $u(t)$ is constant during each sampling interval** (**zero-order hold, ZOH**)

$$u(t) = \bar{u}(k),\ kT_s \leq t < (k+1)T_s$$

- $\bar{x}(k) \triangleq x(kT_s)$ and $\bar{y}(k) \triangleq y(kT_s)$ are the state and the output samples at the k^{th} sampling instant, respectively



EXACT SAMPLING

- Using Lagrange formula, The response of the continuous-time system between $t_0 = kT_s$ and $t = (k+1)T_s$ from $x(t_0) = x(kT_s)$ is

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\sigma)}Bu(\sigma)d\sigma \\ &= e^{A((k+1)T_s - kT_s)}x(kT_s) + \int_{kT_s}^{(k+1)T_s} e^{A((k+1)T_s-\sigma)}Bu(\sigma)d\sigma \end{aligned}$$

- Since the input $u(t)$ is piecewise constant, $u(\sigma) \equiv \bar{u}(k), kT_s \leq \sigma < (k+1)T_s$. By setting $\tau = \sigma - kT_s$ we get

$$x((k+1)T_s) = e^{AT_s}x(kT_s) + \left(\int_0^{T_s} e^{A(T_s-\tau)}d\tau\right) Bu(kT_s)$$

and hence

$$\bar{x}(k+1) = e^{AT_s}\bar{x}(k) + \left(\int_0^{T_s} e^{A(T_s-\tau)}d\tau\right) B\bar{u}(k)$$

which is a linear difference relation between $\bar{x}(k)$ and $\bar{u}(k)$!

EXACT SAMPLING

- The discrete-time system

$$\begin{cases} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k) \\ \bar{y}(k) &= \bar{C}\bar{x}(k) + \bar{D}\bar{u}(k) \end{cases}$$

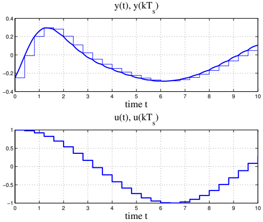
depends on the original continuous-time system through the relations

$$\bar{A} \triangleq e^{AT_s}, \quad \bar{B} \triangleq \left(\int_0^{T_s} e^{A(T_s-\tau)} d\tau\right) B, \quad \bar{C} \triangleq C, \quad \bar{D} \triangleq D$$

- If $u(t)$ is piecewise constant, $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ provides the exact evolution of state and output samples at discrete times kT_s

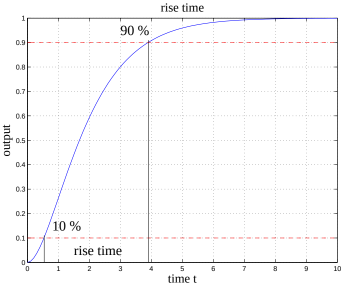
MATLAB

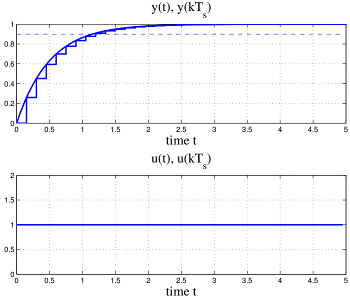
```
sys=ss(A,B,C,D);
sysd=c2d(sys,Ts);
[Ab,Bb,Cb,Db]=ssdata(sysd);
```



CHOICE OF SAMPLING TIME

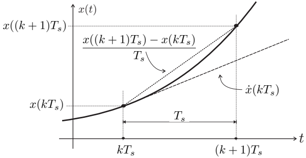
Rule of thumb: $T_s \approx \frac{1}{10}$ of **rise time** = time to move from 10% to 90% of the steady-state value, for input $u(t) \equiv 1, x(0) = 0$





EULER'S FORWARD METHOD

$$\dot{x}(kT_s) \approx \frac{x((k+1)T_s) - x(kT_s)}{T_s}$$



Leonhard Paul Euler
(1707-1783)

- For nonlinear systems $\dot{x}(t) = f(x(t), u(t))$:

$$\bar{x}(k+1) = \bar{x}(k) + T_s f(\bar{x}(k), \bar{u}(k))$$

- For linear systems $\dot{x}(t) = Ax(t) + Bu(t)$:

$$x((k+1)T_s) = (I + T_s A)x(kT_s) + T_s Bu(kT_s)$$

$$\bar{A} \triangleq I + AT_s, \quad \bar{B} \triangleq T_s B, \quad \bar{C} \triangleq C, \quad \bar{D} \triangleq D$$

- $e^{T_s A} = I + T_s A + \ldots + \frac{T_s^n A^n}{n!} + \ldots$ Euler's method $\approx$ exact sampling for $T_s \to 0$

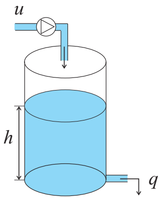
EXAMPLE - HYDRAULIC SYSTEM

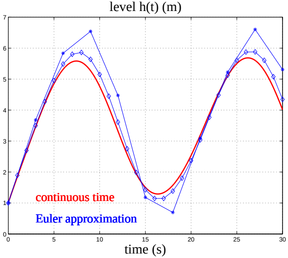
Continuous-time model

$$\begin{cases} \frac{d}{dt}h(t) &=& -\frac{a\sqrt{2g}}{A}\sqrt{h(t)} + \frac{1}{A}u(t) \\ q(t) &=& a\sqrt{2g}\sqrt{h(t)} \end{cases}$$

Discrete-time model

$$\begin{cases} \bar{h}(k+1) &=& \bar{h}(k) - \frac{T_s a\sqrt{2g}}{A}\sqrt{\bar{h}(k)} + \frac{T_s}{A}\bar{u}(k) \\ \bar{q}(k) &=& a\sqrt{2g}\sqrt{\bar{h}(k)} \end{cases}$$





N-STEPS EULER METHOD

- We can obtain the matrices A, B of the discrete-time linearized model while integrating the nonlinear continuous-time dynamic equations $\dot{x} = f(x, u)$

- **N-steps explicit forward Euler method**: given $x(k)$, $u(k)$, execute the following steps
  1. $x = x(k), A = I, B = 0$
  2. for n=1:N do
     - $A \leftarrow (I + \frac{T_s}{N}\frac{\partial f}{\partial x}(x, u(k))A$
     - $B \leftarrow (I + \frac{T_s}{N}\frac{\partial f}{\partial x}(x, u(k))B + \frac{T_s}{N}\frac{\partial f}{\partial u}(x, u(k))A$
     - $x \leftarrow x + \frac{T_s}{N} f(x, u(k))$
  3. end
  4. return $x(k+1) \approx x$ and matrices A, B such that $x(k+1) \approx Ax(k) + Bu(k)$.

- Property: the difference between the state $x(k+1)$ and its approximation x computed by the above iterations satisfies $\|x(k+1) - x)\| = O\left(\frac{T_s}{N}\right)$

- Explicit forward Runge-Kutta 4 method also available

TUSTIN'S DISCRETIZATION METHOD

- Assume $u(k)$ constant within the sampling interval. Given the linear system $\dot{x} = Ax + Bu$, apply the trapezoidal rule to approximate the integral

$$x(k+1) - x(k) = \int_{kT_s}^{(k+1)T_s} \dot{x}(t)dt = \int_{kT_s}^{(k+1)T_s} (Ax(t) + Bu(t))dt$$

$$\approx \frac{T_s}{2}\left(Ax(k) + Bu(k) + Ax(k+1) + Bu(k)\right)$$ (trapezoidal rule)

and therefore

$$(I - \frac{T_s}{2}A)x(k+1) = (I + \frac{T_s}{2})x(k) + T_s Bu(k)$$

$$x(k+1) = \left(I - \frac{T_s}{2}A\right)^{-1}\left(I + \frac{T_s}{2}A\right)x(k) + \left(I - \frac{T_s}{2}A\right)^{-1} T_s Bu(k)$$

- Advantage: simpler to compute than exponential matrix, without too much loss of approximation quality

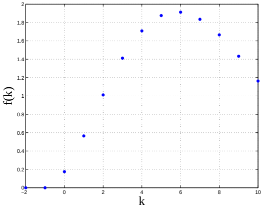
Z-TRANSFORM

Consider a function $f(k)$, $f:\mathbb{Z}\to\mathbb{R}$, $f(k)=0$ for all $k<0$

Definition

The unilateral **Z-transform** of $f(k)$ is the function of the complex variable $z\in\mathbb{C}$ defined by

$$F(z)=\sum_{k=0}^{\infty}f(k)z^{-k}$$



Witold Hurewicz
(1904-1956)

Once $F(z)$ is computed using the series, it's extended to all $z\in\mathbb{C}$ for which $F(z)$ makes sense

Z-transforms convert difference equations into algebraic equations.

EXAMPLES OF Z-TRANSFORMS

- **Discrete impulse**

$$f(k) = \delta(k) \triangleq \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases} \quad \Rightarrow \quad \mathcal{Z}[\delta] = F(z) = 1$$

- **Discrete step**

$$f(k) = 1\!\!1(k) \triangleq \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k \geq 0 \end{cases} \quad \Rightarrow \quad \mathcal{Z}[1\!\!1] = F(z) = \frac{z}{z-1}$$

- **Geometric sequence**

$$f(k) = a^k \, 1\!\!1(k) \;\Rightarrow\; \mathcal{Z}[f] = F(z) = \frac{z}{z-a}$$

PROPERTIES OF Z-TRANSFORMS

- **Linearity**

$$\mathcal{Z}[k_1 f_1(k) + k_2 f_2(k)] = k_1 \mathcal{Z}[f_1(k)] + k_2 \mathcal{Z}[f_2(k)]$$

Example: $f(k) = 3\delta(k) - \frac{5}{2^k}\,\mathbb{I}(t) \Rightarrow \mathcal{Z}[f] = 3 - \frac{5z}{z-\frac{1}{2}}$

- **Forward shift**[3]

$$\mathcal{Z}[f(k+1)\,\mathbb{I}(k)] = z\mathcal{Z}[f] - zf(0)$$

Example: $f(k) = a^{k+1}\,\mathbb{I}(k) \Rightarrow \mathcal{Z}[f] = z\frac{z}{z-a} - z = \frac{az}{z-a}$

[3] z is also called **forward shift operator**

PROPERTIES OF Z-TRANSFORMS

- **Backward shift** or **unit delay** [4]

$$\mathcal{Z}[f(k-1)\,\mathbb{I}(k)] = z^{-1}\mathcal{Z}[f]$$

Example: $f(k) = \mathbb{I}(k-1) \Rightarrow \mathcal{Z}[f] = \frac{z}{z(z-1)}$

- **Multiplication by k**

$$\mathcal{Z}[kf(k)] = -z\frac{d}{dz}\mathcal{Z}[f]$$

Example: $f(k) = k\,\mathbb{I}(k) \Rightarrow \mathcal{Z}[f] = \frac{z}{(z-1)^2}$

[4] z^{-1} is also called **backward shift operator**

DISCRETE-TIME TRANSFER FUNCTION

Apply forward-shift & linearity rules to $x(k+1) = Ax(k) + Bu(k)$, and linearity to $y(k) = Cx(k) + Du(k)$:

$$\begin{array}{rcl} X(z) & = & z(zI-A)^{-1}x_0 + (zI-A)^{-1}BU(z) \\ Y(z) & = & \underbrace{zC(zI-A)^{-1}x_0}_{\text{Z-transform of natural response}} + \underbrace{(C(zI-A)^{-1}B+D)U(z)}_{\text{Z-transform of forced response}} \end{array}$$

Definition

The transfer function of the discrete-time linear system (A, B, C, D) is

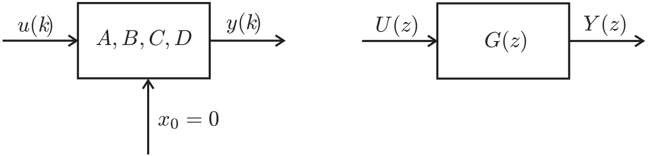
$$G(z) = C(zI-A)^{-1}B + D$$

that is the ration between the Z-transform $Y(z)$ of the output and the Z-transform $U(z)$ of the input signals for the initial state $x_0 = 0$

MATLAB

```
»sys=ss(A,B,C,D,Ts); »G=tf(sys)
```

DISCRETE-TIME TRANSFER FUNCTION



Example: The linear system

$$\begin{cases} x(k+1) & = & \begin{bmatrix} 0.5 & 1 \\ 0 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) & = & \begin{bmatrix} 1 & -1 \end{bmatrix} x(k) \end{cases}$$

with sampling time $T_s = 0.1$ s has the transfer function

$$G(z) = \frac{-z + 1.5}{z^2 - 0.25}$$

Note: Even for discrete-time systems, the transfer function does not depend on the input $u(k)$. It's only a property of the linear system

MATLAB

```
sys=ss([0.5 1;
  0 -0.5],[0;1],[1 -1],0,0.1);
G=tf(sys)

Transfer function:
-z + 1.5
----------
s^2 - 0.25
```

DIFFERENCE EQUATIONS

- Consider the n^{th}-order difference equation forced by u

$$\begin{aligned} & a_n y(k-n) + a_{n-1} y(k-n+1) + \cdots + a_1 y(k-1) + y(k) \\ & = b_n u(k-n) + \cdots + b_1 u(k-1) \end{aligned}$$

- For zero initial conditions we get the transfer function

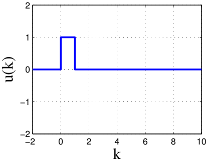
$$\begin{aligned} G(z) &= \frac{b_n z^{-n} + b_{n-1} z^{-n+1} + \cdots + b_1 z^{-1}}{a_n z^{-n} + a_{n-1} z^{-n+1} + \cdots + a_1 z^{-1} + 1} \\ &= \frac{b_1 z^{n-1} + \cdots + b_{n-1} z + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n} \end{aligned}$$

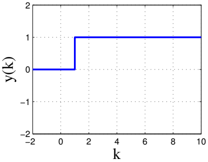
IMPULSE RESPONSE

- Consider the impulsive input $u(k) = \delta(k), U(z) = 1$. The corresponding output $y(k)$ is called **impulse response**
- The Z-transform of $y(k)$ is $Y(z) = G(z) \cdot 1 = G(z)$
- Therefore the impulse response coincides with the **inverse Z-transform** $g(k)$ of the transfer function $G(z)$

Example (integrator:)

$$\begin{aligned} u(k) &= \delta(k) \\ y(k) &= \mathcal{Z}^{-1}\left[\frac{1}{z-1}\right] = \mathbb{1}(k-1) \end{aligned}$$





POLES, EIGENVALUES, MODES

- Linear discrete-time system

$$\left\{\begin{array}{rcl} x(k+1) & = & Ax(k)+Bu(k) \\ y(k) & = & Cx(k)+Du(k) \end{array}\right. \qquad G(z)=C(zI-A)^{-1}B+D \triangleq \frac{N_G(z)}{D_G(z)}$$

$$x(0)=0$$

- Use the adjugate matrix to represent the inverse of $zI-A$

$$C(zI-A)^{-1}B+D=\frac{C\,\mathrm{Adj}(zI-A)B}{\det(zI-A)}+D$$

- The denominator $D_G(z)=\det(zI-A)$!

The poles of $G(z)$ coincide with the eigenvalues of A

- Well, not always ... (as in continuous time)

STEADY-STATE SOLUTION AND DC GAIN

- Let A asymptotically stable ($|\lambda_i| < 1$). The natural response vanishes asymptotically
- Assume constant $u(k) \equiv u_r, \forall k \in \mathbb{N}$. What is the asymptotic value $x_r = \lim_{k\to\infty} x(k)$?

  Impose $x_r(k+1) = x_r(k) = Ax_r + Bu_r$ and get $x_r = (I - A)^{-1} Bu_r$

  The corresponding **steady-state** output $y_r = Cx_r + Du_r$ is

$$y_r = \underbrace{(C(I - A)^{-1}B + D)}_{\text{DC gain}} u_r$$

- Cf. final value theorem:

$$\begin{aligned} y_r &= \lim_{k\to+\infty} y(k) = \lim_{z\to 1} (z-1)Y(z) = \lim_{z\to 1} (z-1)G(z)U(z) \\ &= \lim_{z\to 1} (z-1)G(z)\frac{u_r z}{z-1} = G(1)u_r = (C(I-A)^{-1}B + D)u_r \end{aligned}$$

- $G(1)$ is called the **DC gain** of the system

Example - student population dynamics

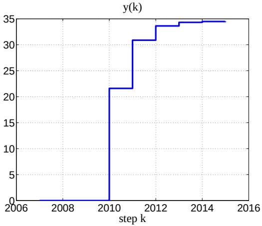
- Recall student population dynamics

$$\begin{cases} x(k+1) &= \begin{bmatrix} .2 & 0 & 0 \\ .6 & .15 & 0 \\ 0 & .8 & .08 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & .9 \end{bmatrix} x(k) \end{cases}$$

- DC gain:

$$\begin{bmatrix} 0 & 0 & .9 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .2 & 0 & 0 \\ .6 & .15 & 0 \\ 0 & .8 & .08 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \approx 0.69$$

- Transfer function: $G(z) = \frac{0.432}{z^3 - 0.43z^2 + 0.058z - 0.0024}$, $G(1) \approx 0.69$



MATLAB

```
»A=[b1 0 0; a1 b2 0; 0 a2 b3];
»B=[1;0;0];
»C=[0 0 a3];
»D=[0];
»sys=ss(A,B,C,D,1);
»dcgain(sys)

ans =

   0.6905
```

- For $u(k) \equiv 50$ students enrolled steadily, $y(k) \to 0.69 \cdot 50 \approx 34.5$ graduate

CLOSED-LOOP CONTROL

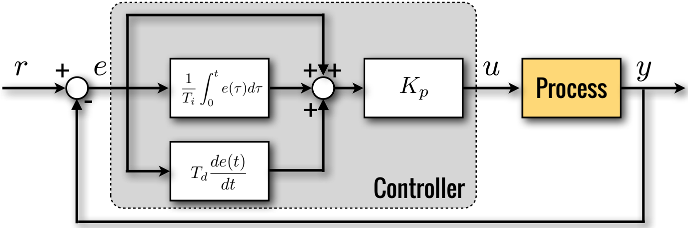
- **PID (proportional integrative derivative) controllers** are the most used controllers in industrial automation since the ’30s

$$u(t) = K_p \left[e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right]$$

where $e(t) = r(t) - y(t)$ is the tracking error

- Initially constructed by analog electronic components, today they are implemented digitally
  - ad hoc digital devices
  - just few lines of C code included in the control unit

PID PARAMETERS



- K_p is the **controller gain**, determining the "aggressiveness" of the controller
- T_i is the **reset time**, determining the weight of the integral action. The integral action guarantees that in steady-state $y(t) = r(t)$
- T_d is the **derivative time**. The term $e(t) + T_d\frac{de(t)}{dt}$ provides a "prediction" of the tracking error at time $t + T_d$
- We call the controller P, PD, PI, or PID depending on the feedback terms included in the control law

STRUCTURE OF PID CONTROLLER

- In practice one implements the following version of the PID controller

$$u(t) = K_p\Big[\underbrace{br(t) - y(t)}_{\text{proportional action}} + \underbrace{\frac{1}{sT_i}\int_0^t (r(\tau) - y(\tau))d\tau}_{\text{integral action}} + \underbrace{d(t)}_{\text{derivative action}}\Big]$$

$$d(t) + \frac{T_d}{N}\dot{d}(t) = -T_d\dot{y}(t)$$

- the reference signal $r(t)$ is not included in the derivative term ($r(t)$ may have abrupt changes)
- the proportional action $K_p(br(t) - y(t)$ only uses a fraction $b \leq 1$ of the reference signal $r(t)$
- the derivative term $d(t)$ is a filtered version of $\dot{y}(t)$

- In digital (=discrete-time) form with sampling time T_s, the PID controller takes the following form

$$u(k) = P(k) + I(k) + D(k)$$

$$P(k) = K_p(br(k) - y(k))$$

$$I(k+1) = I(k) + \frac{K_p T_s}{T_i}(r(k) - y(k))$$ forward differences

$$D(k) = \frac{T_d}{T_d + NT_s} D(k-1) - \frac{K_p T_d N}{T_d + NT_s}(y(k) - y(k-1))$$

backward differences

PID CONTROLLER: PROS AND CONS

- Very simple to implement, only 3 parameters to calibrate
- It only requires the measurement of the output signal $y(t)$
- The control law does not exploit the knowledge of the model of the process
- Achievable closed-loop performance is limited

STATE-FEEDBACK CONTROL

REACHABILITY ANALYSIS

- Consider the linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

  with $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$ and initial condition $x(0) = x_0 \in \mathbb{R}^n$

- The solution is $x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^j B u(k-1-j)$

Definition

The system $x(k+1) = Ax(k) + Bu(k)$ is **(completely) reachable** if $\forall x_1, x_2 \in \mathbb{R}^n$ there exist $k \in \mathbb{N}$ and $u(0), u(1), \ldots, u(k-1) \in \mathbb{R}^m$ such that

$$x_2 = A^k x_1 + \sum_{j=0}^{k-1} A^j B u(k-1-j)$$

- In simple words: a system is completely reachable if from any state x_1 we can reach any state x_2 at some time k, by applying a suitable input sequence

REACHABILITY

- Determine a sequence of n inputs transferring the state vector from x_1 to x_2 after n steps

$$\underbrace{x_2 - A^n x_1}_{X} = \underbrace{\left[\begin{array}{cccc} B & AB & \ldots & A^{n-1}B \end{array}\right]}_{R} \underbrace{\left[\begin{array}{c} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{array}\right]}_{U}$$

- This is equivalent to solve with respect to U the linear system of equations

$$RU = X$$

- Matrix $R \in \mathbb{R}^{n \times nm}$ is called the **reachability matrix** of the system
- A solution U exists if and only if $X \in \mathrm{Im}(R)$
  (Rouché-Capelli theorem: a solution exists $\Leftrightarrow \mathrm{rank}([R\ X]) = \mathrm{rank}(R)$)

Theorem

The system (A, B) is completely reachable $\Leftrightarrow \text{rank}(R) = n$

Proof:

($\Rightarrow$) Assume (A, B) reachable, choose $x_1 = 0$ and $x_2 = x$. Then $\exists k \geq 0$ such that

$$x = \sum_{j=0}^{k-1} A^j Bu(k-1-j)$$

If $k \leq n$, then clearly $x \in \text{Im}(R)$. If $k > n$, by Cayley-Hamilton theorem we have again $x \in \text{Im}(R)$. Since x is arbitrary, $\text{Im}(R) = \mathbb{R}^n$, so $\text{rank}(R) = n$.

($\Leftarrow$) If $\text{rank}(R) = n$, then $\text{Im}(R) = \mathbb{R}^n$. Let $X = x_2 - A^n x_1$ and $U = [\ u(n-1)' \ \ldots \ u(1)' \ u(0)']'$. The system $X = RU$ can be solved with respect to $U, \forall X$, so any state x_1 can be transferred to x_2 in $k = n$ steps. Therefore, the system (A, B) is completely reachable.

MINIMUM-ENERGY CONTROL

- Let (A, B) reachable and consider steering the state from $x(0) = x_1$ into $x(k) = x_2, k > n$

$$\underbrace{x_2 - A^k x_1}_{X} = \underbrace{\begin{bmatrix} B & AB & \dots & A^{k-1}B \end{bmatrix}}_{R_k} \underbrace{\begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}}_{U}$$

($R_k \in \mathbb{R}^{n \times km}$ is the reachability matrix for k steps)

- Since $\mathrm{rank}(R_k) = \mathrm{rank}(R) = n, \forall k > n$ (Cayley-Hamilton), we get $\mathrm{rank}\, R_k = \mathrm{rank}[R_k\ X] = n$
- Hence the system $X = R_k U$ admits solutions U

Problem

Determine the input sequence $\{u(j)\}_{j=0}^{k-1}$ that brings the state from $x(0) = x_1$ to $x(k) = x_2$ with minimum energy $\frac{1}{2}\sum_{j=0}^{k-1} \|u(j)\|^2 = \frac{1}{2}U'U$

MINIMUM-ENERGY CONTROL

- The problem is equivalent to finding the solution U of the system of equations

$$X = R_k U$$

with minimum norm $\|U\|$

- We must solve the optimization problem

$$U^* = \arg\min \frac{1}{2}\|U\|^2 \quad \text{subject to} \quad X = R_k U$$

- Let's apply the method of Lagrange multipliers:

$$\mathcal{L}(U,\lambda) = \frac{1}{2}\|U\|^2 + \lambda'(X - R_k U)$$ Lagrangian function

$$\frac{\partial \mathcal{L}}{\partial U} = U - R_k'\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = X - R_k U = 0$$

$$\Rightarrow \quad U^* = \underbrace{R_k'(R_k R_k')^{-1}}_{R_k^{\#}} \cdot X$$

$R_k^{\#}$ = pseudoinverse matrix

MATLAB
U=pinv(Rk)*X

- Note that $R_k R_k'$ is invertible because $\mathrm{rank}(R_k) = \mathrm{rank}(R) = n, \forall k \geq n$

CONTROLLABILITY

- If the system is completely reachable, we have seen that we can bring the state vector from any value $x(0) = x_1$ to any other value $x(n) = x_2$

- Let's focus on the subproblem of determining a finite sequence of inputs that brings the state to the final value $x(n) = 0$

Definition

A system $x(k+1) = Ax(k) + Bu(k)$ is **controllable** to the origin in k steps if $\forall x_0 \in \mathbb{R}^n$ there exists a sequence $u(0), u(1), \ldots, u(k-1) \in \mathbb{R}^m$ such that $0 = A^k x_0 + \sum_{j=0}^{k-1} A^j Bu(k-1-j)$

- Controllability is a weaker condition than reachability

CONTROLLABILITY, STABILIZABILITY

- The linear system of equations

$$-A^n x_0 = \underbrace{\left[\, B\ AB\ \ldots\ A^{n-1}B \,\right]}_{R} \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}$$

admits a solution if and only if $A^n x_0 \in \mathrm{Im}(R), \forall x_0 \in \mathbb{R}^n$

Theorem

The system is controllable to the origin (in n steps) if and only if

$$\mathrm{Im}(A^n) \subseteq \mathrm{Im}(R)$$

Definition

A linear system $x(k+1) = Ax(k) + Bu(k)$ is called **stabilizable** if can be driven asymptotically to the origin

- Stabilizability is a weaker condition than controllability

REACHABILITY ANALYSIS OF CONTINUOUS-TIME SYSTEMS

- Similar definitions of reachability, controllability, and stabilizability can be given for continuous-time systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- No distinction between controllability and reachability in continuous-time (because no finite-time convergence of modal response exists)

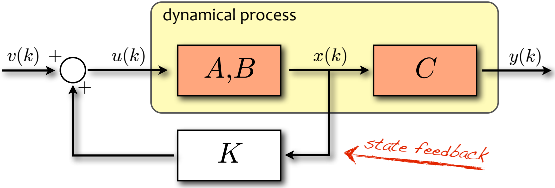
- Reachability matrix and canonical reachability decomposition are identical to discrete-time

- $\mathrm{rank}\, R = n$ is also a necessary and sufficient condition for reachability

- A_{uc} asymptotically stable (all eigenvalues with negative real part) is also a necessary and sufficient condition for stabilizability

STABILIZATION BY STATE FEEDBACK

- **Main idea**: design a device that makes the process (A, B, C) asymptotically stable by manipulating the input u to the process



- If measurements of the state vector are available, we can set

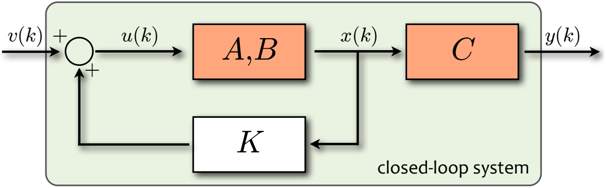
$$u(k) = k_1 x_1(k) + k_2 x_2(k) + \ldots + k_n x_n(k) + v(k)$$

- $v(k)$ is an exogenous signal exciting the closed-loop system

Problem

Find a feedback gain $K = [k_1 \; k_2 \; \ldots \; k_n]$ that makes the closed-loop system asymptotically stable.

STABILIZATION BY STATE FEEDBACK



- Let $u(k) = Kx(k) + v(k)$. The overall system is

$$\begin{aligned} x(k+1) &= (A+BK)x(k) + Bv(k) \\ y(k) &= (C+DK)x(k) + Dv(k) \end{aligned}$$

Theorem

(A, B) "reachable" ($\text{rank}\left[\, B \;\, AB \;\, \ldots \;\, A^{n-1}B \,\right] = n$) $\Rightarrow$ the eigenvalues of $(A+BK)$ can be decided **arbitrarily**.

Fact

(A, B) reachable $\Leftrightarrow$ (A, B) is algebraically equivalent to a pair $(\tilde{A}, \tilde{B})$ in **controllable canonical form**

$$\tilde{A} = \begin{bmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The transformation matrix T such that $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$ is

$$T = [B\ AB\ \dots A^{n-1}B] \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_2 & a_3 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

where $a_1, a_2, \dots, a_{n-1}$ are the coefficients of the characteristic polynomial

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = \det(\lambda I - A)$$

- Let (A, B) reachable and assume $m = 1$ (single input)
- Characteristic polynomials:

$$\begin{aligned} p_A(\lambda) &= \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 && \text{(open-loop eigenvalues)} \\ p_d(\lambda) &= \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0 && \text{(desired closed-loop eigenvalues)} \end{aligned}$$

- Let (A, B) be in controllable canonical form

$$A = \begin{bmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ -a_0 & -a_1 & \ldots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- As $K = [k_1 \ \ldots \ k_n]$, we have

$$A + BK = \begin{bmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ -(a_0 - k_1) & -(a_1 - k_2) & \ldots & -(a_{n-1} - k_n) \end{bmatrix}$$

- The characteristic polynomial of $A + BK$ is therefore

$$\lambda^n + (a_{n-1} - k_n)\lambda^{n-1} + \ldots + (a_1 - k_2)\lambda + (a_0 - k_1)$$

- To match $p_d(\lambda)$ we impose

$$a_0 - k_1 = d_0 \ , \ a_1 - k_2 = d_1 \ , \ \ldots \ , \ a_{n-1} - k_n = d_{n-1}$$

Procedure

If (A, B) is in controllable canonical form, the feedback gain

$$K = \left[\, a_0 - d_0 \; a_1 - d_1 \; \ldots \; a_{n-1} - d_{n-1} \, \right]$$

makes $p_d(\lambda)$ the characteristic polynomial of $(A + BK)$

- If (A, B) is not in controllable canonical form we must set

$$\begin{aligned}\tilde{K} &= \begin{bmatrix} a_0 - d_0 & a_1 - d_1 & \dots & a_{n-1} - d_{n-1} \end{bmatrix} \\ K &= \tilde{K} T^{-1} \quad \leftarrow \text{don't invert } T\text{, solve instead } T'K' = \tilde{K}' \text{ w.r.t. } K' \text{ !}\end{aligned}$$

where

$$T = R \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_2 & a_3 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

- Explanation: a matrix M and $T^{-1}MT$ have the same eigenvalues

$$\begin{aligned}\det(\lambda I - T^{-1}MT) &= \det(T^{-1}T\lambda - T^{-1}MT) = \det(T^{-1}) \det(\lambda I - M) \\ &\quad \cdot \det(T) = \det(\lambda I - M)\end{aligned}$$

- Since $(\tilde{A} + \tilde{B}\tilde{K}) = T^{-1}AT + T^{-1}BKT = T^{-1}(A + BK)T$, it follows that $(\tilde{A} + \tilde{B}\tilde{K})$ and $(A + BK)$ have the same eigenvalues

ACKERMANN'S FORMULA

- Let (A, B) reachable and assume $m = 1$ (single input)
- Characteristic polynomials:

$$\begin{aligned} p_A(\lambda) &= \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \text{ (open-loop eigenvalues)} \\ p_d(\lambda) &= \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0 \text{ (desired closed-loop eigenvalues)} \end{aligned}$$

- Let $p_d(A) = A^n + d_{n-1}A^{n-1} + \ldots + d_1 A + d_0 I$ ← (This is $n \times n$ matrix !)

Ackermann's formula

$$K = -[0 \ \ldots \ 0\, 1][B \ AB \ \ldots \ A^{n-1}B]^{-1} p_d(A)$$

MATLAB

```
K=-acker(A,B,P);
K=-place(A,B,P);
```

where $P = [\lambda_1 \lambda_2 \ldots \lambda_n]$ are the desired closed-loop poles

ZEROS OF CLOSED-LOOP SYSTEM

Fact

The zeros of the system are the same under state feedback: $N_K(z) = N(z)$

- Example for $x \in \mathbb{R}^3$: change the coordinates to canonical reachability form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; K = \begin{bmatrix} k_3 & k_2 & k_1 \end{bmatrix}$$

- Compute $N(z)$

$$\mathrm{Adj}(zI - A)B = \begin{bmatrix} z^2 + a_1 z + a_2 & z + a_1 & 1 \\ -a_3 & z(z + a_1) & z \\ -a_3 z & -a_2 z - a_3 & z^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}$$

- $\mathrm{Adj}(zI - A)B$ does not depend on the coefficients a_1, a_2, a_3. So also $\mathrm{Adj}(zI - A - BK)B$ does not depends on $a_1 - k_1, a_2 - k_2, a_3 - k_3$
- $N(z) = C\,\mathrm{Adj}(zI - A)B = C\,\mathrm{Adj}(zI - A - BK)B = N_K(z), \forall K' \in \mathbb{R}^n$

EXAMPLE - STUDENT POPULATION DYNAMICS

- The open-loop poles are $(0.8, 0.15, 0.2)$
- Say we want to place the closed-loop poles in $(0.1 \pm 0.4j, 0.1)$ by setting

$$u(k) = Kx(k) + Hr(k)$$

where $r(k)$ is the desired reference signal

- First, design K by pole placement:

MATLAB
```
K=-place(A,B,[.1+.4*j,.1-.4*j,.1])
```

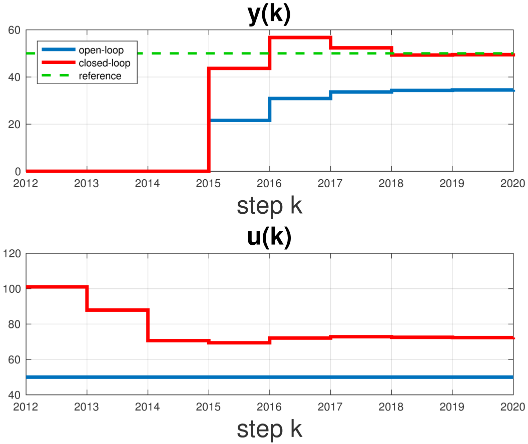
- Then choose H such that the DC-gain from r to y is 1:

MATLAB
```
sys_cl=ss(A+B*K,B,C+D*K,D,1);
dc_cl=dcgain(sys_cl);
H=1/dc_cl;
```

- We get $K = [-0.1300\ -0.2698\ 0.0067]$, $H = 2.0208$

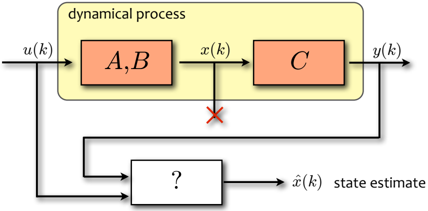
EXAMPLE - STUDENT POPULATION DYNAMICS

- Compare open-loop vs. closed-loop response



STATE ESTIMATION

OBSERVABILITY



- Implementing a state feedback controller $u(k) = Kx(k)$ requires the entire state vector $x(k)$
- **Problem:** often sensors only provide the measurements of output $y(k)$
- **Idea**: is it possible to estimate the state x by measuring only the output y and knowing the applied input u ?
- **Observability** analysis addresses this problem, telling us when and how the state estimation problem can be solved

OBSERVABILITY

- Consider $$\begin{cases} x(k+1) & = & Ax(k)+Bu(k) \\ y(k) & = & Cx(k)+Du(k) \end{cases}$$
  with $x\in\mathbb{R}^n, u\in\mathbb{R}, y\in\mathbb{R}$ and initial condition $x(0)=x_0\in\mathbb{R}^n$ ([5])
- The solution for the output is

$$y(k,x_0,u(\cdot)) = CA^k x_0 + \sum_{j=0}^{k-1} CA^j Bu(k-1-j) + Du(k)$$

Definition

The pair of states $x_1 \neq x_2 \in \mathbb{R}^n$ is called **indistinguishable** from the output $y(\cdot)$ if for any input sequence $u(\cdot)$

$$y(k,x_1,u(\cdot)) = y(k,x_2,u(\cdot)), \forall k \geq 0$$

A linear system is called **(completely) observable** if no pair of states are indistinguishable from the output

[5] Everything here can be easily generalized to multivariable systems $u\in\mathbb{R}^m, y\in\mathbb{R}^p$

OBSERVABILITY

- Consider the problem of reconstructing the initial condition x_0 from n output measurements, applying a known input sequence

$$\begin{array}{rcl} y(0) & = & Cx_0 + Du(0) \\ y(1) & = & CAx_0 + CBu(0) + Du(1) \\ & \vdots & \\ y(n-1) & = & CA^{n-1}x_0 + \sum_{j=1}^{n-2} CA^jBu(n-2-j) + Du(n-1) \end{array}$$

- Define

$$\Theta = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\text{This is a } n \times n \text{ matrix}} \qquad Y = \underbrace{\begin{bmatrix} y(0) - Du(0) \\ y(1) - CBu(0) - Du(1) \\ \vdots \\ y(n-1) - \sum_{j=1}^{n-2} CA^jBu(n-2-j) - Du(n-1) \end{bmatrix}}_{\text{This is an } n\text{-th dimensional vector}}$$

OBSERVABILITY

- The initial state x_0 is determined by solving the linear system

$$Y = \Theta x_0$$

  The matrix $\Theta \in \mathbb{R}^{n \times n}$ is called the **observability matrix** of the system

- If we assume perfect knowledge of the output (i.e., no noise on output measurements), we can always solve the system $Y = \Theta x_0$. In particular:
  - There is only one solution if $\mathrm{rank}(\Theta) = n$
  - There exist infinite solutions if $\mathrm{rank}(\Theta) < n$. In this case, all solutions are given by $x_0 + \ker(\Theta)$, where x_0 is any particular solution of the system

- Knowing x_0, we know $x(k) = A^k x_0 + \sum_{i=0}^{k-1} A^i B u(k-1-i)$ for all $k \geq 0$

OBSERVABILITY

- The system of equations $\Theta x_0 = Y$ has a solution if and only if

$$\text{rank}(\Theta) = \text{rank}([\Theta\, Y]) \qquad \text{(Rouché-Capelli Theorem)}$$

- Because we have $\Theta \in \mathbb{R}^{n \times n}$, if $\text{rank}(\Theta) = n \Rightarrow \text{rank}([\Theta\, Y]) = n$ for each Y

- The solution is unique if and only if $\text{rank}(\Theta) = n$

- Since the input $u(k)$ influences only the known vector Y, the solvability of the system $\Theta x_0 = Y$ is independent from $u(k)$

- Then, for linear systems the observability property does not depend on the input signal $u(\cdot)$, it only depends on matrix Θ (i.e., on A and C)

- We can study the observability properties of the system for $u(k) \equiv 0$

OBSERVABILITY

Theorem

A linear system is observable if and only if $\text{rank}(\Theta) = n$

- As the observability property of a system depends only on matrices A and C, we call a pair (A, C) **observable** if

$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$$

- It can be proved that $\ker(\Theta)$ is the set of states $x \in \mathbb{R}^n$ that are indistinguishable from the origin

$$y(k, x, u(\cdot)) = y(k, 0, u(\cdot)),\ \forall k \geq 0$$

  for any input sequence $u(\cdot)$
- Since $\ker(\Theta) = \{0\}$ if and only if $\text{rank}(\Theta) = n$, a system is observable if and only if there are no states that are indistinguishable from the origin $x = 0$

RECONSTRUCTABILITY

- Under observability assumptions, we just saw that it is possible to determine the initial condition x_0 from n input/output measurements

$$x(0) = \Theta^{-1} Y$$

- To close the control loop at time k it is enough to know the current $x(k)$

- If the initial condition $x(0)$ is known, it is possible to calculate $x(k)$ as

$$x(k) = A^k \Theta^{-1} Y + \sum_{i=0}^{k-1} A^i B u(k-1-i)$$

- **Question:** Can we determine the current state $x(k)$ even if the system is not completely observable?

RECONSTRUCTABILITY

Definition

A linear system $x(k+1) = Ax(k) + Bu(k)$ is called **reconstructable** in k steps if, for each initial condition x_0, $x(k)$ is uniquely determined by $\{u(j), y(j)\}_{j=0}^{k-1}$

The solutions of the system

$$Y_k \triangleq \begin{bmatrix} y(0) - Du(0) \\ y(1) - CBu(0) - Du(1) \\ \vdots \\ y(k-1) - \sum_{j=1}^{k-2} CA^j Bu(k-2-j) + Du(k-1) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}}_{\Theta_k} x$$

are given by $x = x_0 + \ker(\Theta_k)$, where x_0 is the "true" (unknown) initial state

RECONSTRUCTABILITY

- Let x_0 be the initial (unknown) "true" state, and $x = x_0 + \bar{x}$ be a generic initial state, where $\bar{x} \in \ker(\Theta_k)$. An estimation $\hat{x}(k)$ of the current state $x(k)$ is

$$\hat{x}(k) = A^k x_0 + A^k \bar{x} + \sum_{j=1}^{k-1} A^j Bu(k-1-j)$$

- $\hat{x}(k)$ coincides with $x(k)$ if and only if $\bar{x} \in \ker(A^k)$. Because this must hold for any $\bar{x} \in \ker(\Theta_k)$, we have the following

Lemma

A system is reconstructable in k steps if and only if $\ker(\Theta_k) \subseteq \ker(A^k)$

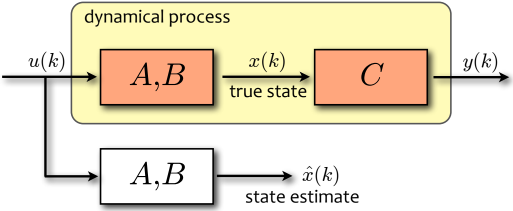
Definition

A system is called **detectable** if it is reconstructable asymptotically for $t \to +\infty$

State estimation problem

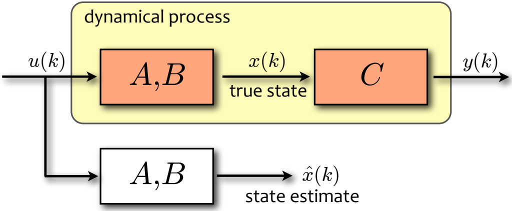
At each time k construct an estimate $\hat{x}(k)$ of the state $x(k)$, by only measuring the output $y(k)$ and input $u(k)$.

- **Open-loop observer**: Build an artificial copy of the system, fed in parallel by with the same input signal $u(k)$



- The "copy" is a numerical simulator $\hat{x}(k+1) = A\hat{x}(k) + Bu(k)$ reproducing the behavior of the real system

OPEN-LOOP OBSERVER



- The dynamics of the real system and of the numerical copy are

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) \end{aligned}$$

True process

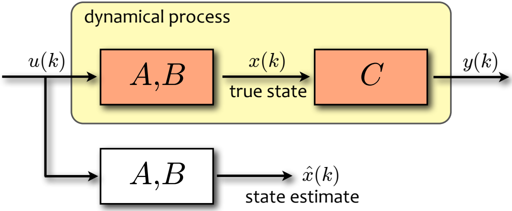
Numerical copy

- The dynamics of the **estimation error** $\tilde{x}(k) = x(k) - \hat{x}(k)$ are

$$\tilde{x}(k+1) = Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) = A\tilde{x}(k)$$

and then $\tilde{x}(k) = A^k(x(0) - \hat{x}(0))$

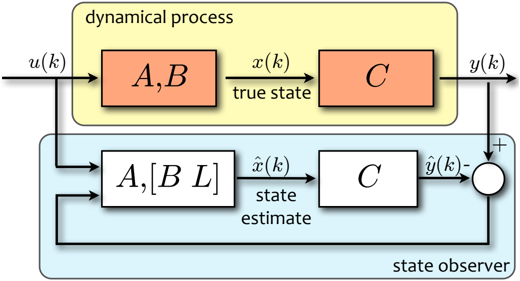
OPEN-LOOP OBSERVER



The estimation error is $\tilde{x}(k) = A^k(x(0) - \hat{x}(0))$. This is not ideal, because

- The dynamics of the estimation error are fixed by the eigenvalues of A and cannot be modified
- The estimation error vanishes asymptotically if and only if A is asymptotically stable
- Note that we are not exploiting $y(k)$ to compute the state estimate $\hat{x}(k)$!

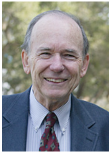
LUENBERGER OBSERVER



- **Luenberger observer**: Correct the estimation equation with a feedback from the estimation error $y(k) - \hat{y}(k)$

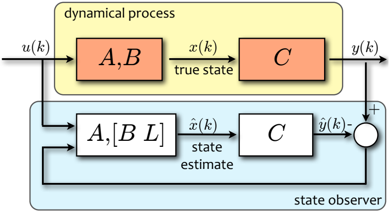
$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + \underbrace{L(y(k) - C\hat{x}(k))}_{\text{feedback on estimation error}}$$

where $L \in \mathbb{R}^{n \times p}$ is the **observer gain**



David G. Luenberger
(1937–)

LUENBERGER OBSERVER



- The dynamics of the state estimation error $\tilde{x}(k) = x(k) - \hat{x}(k)$ is

$$\begin{aligned}\tilde{x}(k+1) &= Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) - L[y(k) - C\hat{x}(k)] \\ &= (A - LC)\tilde{x}(k)\end{aligned}$$

and then $\tilde{x}(k) = (A - LC)^k (x(0) - \hat{x}(0))$

- Same idea for continuous-time systems $\dot{x}(t) = Ax(t) + Bu(t)$

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + L[y(t) - C\hat{x}(t)]$$

The dynamics of the state estimation error are $\frac{d\tilde{x}(t)}{dt} = (A - LC)\tilde{x}(t)$

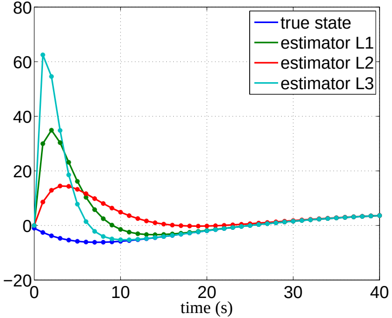
EIGENVALUE ASSIGNMENT OF STATE OBSERVER

Theorem

If the pair (A, C) is "observable" (= (A', C') "reachable"), then the eigenvalues of $(A - LC)$ can be placed arbitrarily.

MATLAB
L=acker(A',C',P)';
L=place(A',C',P)';

where $P = [\lambda_1 \lambda_2 \ldots \lambda_n]$ = desired observer eigenvalues



response from initial conditions $x(0) = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right], \hat{x}(0) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ for $u(k) \equiv 0.1$ for different choices of the observer poles

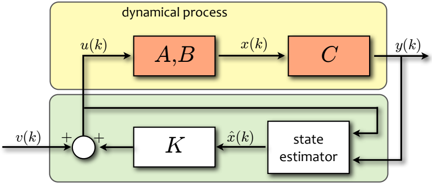
DYNAMIC COMPENSATORS

POTENTIAL ISSUES IN STATE FEEDBACK CONTROL

- Measuring the entire state vector may be too expensive (many sensors)
- It may be even impossible (high temperature, high pressure, inaccessible environment)

Can we use the estimate $\hat{x}(k)$ instead of $x(k)$ to close the loop ?

DYNAMIC COMPENSATOR



- Assume the open-loop system is completely observable and reachable
- Construct the linear state observer

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))$$

- Set $u(k) = K\hat{x}(k) + v(k)$
- The dynamics of the error estimate $\tilde{x}(k) = x(k) - \hat{x}(k)$ is

$$\tilde{x}(k+1) = Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) + L(Cx(k) - C\hat{x}(k)) = (A - LC)\tilde{x}(k)$$

The error estimate does not depend on the feedback gain K !

CLOSED-LOOP DYNAMICS

- Let's combine the dynamics of the system, observer, and feedback gain

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) \\ \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k)) \\ u(k) &= K\hat{x}(k) + v(k) \\ y(k) &= Cx(k) \end{cases}$$

- Take $x(k)$, $\tilde{x}(k)$ as state components of the closed-loop system

$$\begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$$

(it is indeed a change of coordinates)

- The closed-loop dynamics is

$$\begin{cases} \begin{bmatrix} x(k+1) \\ \tilde{x}(k+1) \end{bmatrix} &= \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(k) \\ y(k) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix} \end{cases}$$

CLOSED-LOOP DYNAMICS

- The transfer function from $v(k)$ to $y(k)$ is

$$\begin{aligned} G(z) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} zI-A-BK & BK \\ 0 & zI-A+LC \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} (zI-A-BK)^{-1} & \star \\ 0 & (zI-A+LC)^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= C(zI-A-BK)^{-1}B = \frac{N(z)}{D_K(z)} \end{aligned}$$

- Even if we substituted $x(k)$ with $\hat{x}(k)$, the input-output behavior of the closed-loop system didn't change !

The closed-loop poles can be assigned arbitrarily using **dynamic** output feedback, as in the state feedback case

The closed-loop transfer function does not depend on the observer gain L

SEPARATION PRINCIPLE

Separation principle

The design of the control gain K and of the observer gain L can be done independently

- Watch out ! $G(z) = C(zI - A - BK)^{-1}B$ only represents the I/O (=input/output) behavior of the closed-loop system
- The complete set of poles of the closed-loop system are given by

$$\det(zI - \left[\begin{smallmatrix} A+BK & -BK \\ 0 & A-LC \end{smallmatrix}\right]) = \det(zI - A - BK)\det(zI - A + LC) = D_K(z)D_L(z)$$

- A zero/pole cancellation of the observer poles has occurred:

$$G(z) = \begin{bmatrix} C & 0 \end{bmatrix} (zI - \left[\begin{smallmatrix} A+BK & -BK \\ 0 & A-LC \end{smallmatrix}\right])^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} = \frac{N(z)D_L(z)}{D_K(z)D_L(z)}$$

TRANSIENT EFFECTS OF THE ESTIMATOR GAIN

- L has an effect on the natural response of the system !
- To see this, consider the effect of a nonzero initial condition $\left[\begin{smallmatrix} x(0) \\ \tilde{x}(0) \end{smallmatrix}\right]$ for $v(k) \equiv 0$

$$\begin{aligned}
y(0) &= Cx(0) \\
y(1) &= \begin{bmatrix} C & 0 \end{bmatrix} \left[\begin{smallmatrix} A+BK & -BK \\ 0 & A-LC \end{smallmatrix}\right] \left[\begin{smallmatrix} x(0) \\ \tilde{x}(0) \end{smallmatrix}\right] \\
&= \begin{bmatrix} C & 0 \end{bmatrix} \left[\begin{smallmatrix} (A+BK)x(0)-BK\tilde{x}(0) \\ (A-LC)\tilde{x}(0) \end{smallmatrix}\right] = C(A+BK)x(0) - CBK\tilde{x}(0) \\
y(2) &= \begin{bmatrix} C & 0 \end{bmatrix} \left[\begin{smallmatrix} A+BK & -BK \\ 0 & A-LC \end{smallmatrix}\right] \left[\begin{smallmatrix} x(1) \\ \tilde{x}(1) \end{smallmatrix}\right] \\
&= C(A+BK)x(1) - CBK\tilde{x}(1) \\
&= C(A+BK)^2x(0) - C(A+BK)BK\tilde{x}(0) - CBK(A-\mathbf{L}C)\tilde{x}(0)
\end{aligned}$$

- If $\tilde{x}(0) \neq 0$, L has an effect during the transient !

CHOOSING THE ESTIMATOR GAIN

- Intuitively, if $\hat{x}(k)$ is a poor estimate of $x(k)$ then the control action will also be poor

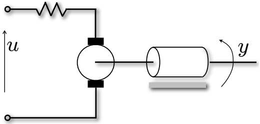
Rule of thumb: place the observer poles ≈ 10 times faster than the controller poles

- Optimal methods exist to choose the observer poles (Kalman filter)

- Fact: The choice of L is very important for determining the sensitivity of the closed-loop system with respect to input and output noise

EXAMPLE: CONTROL OF A DC MOTOR

$$\frac{d^3y}{dt} + \beta\frac{d^2y}{dt} + \alpha\frac{dy}{dt} = Ku$$



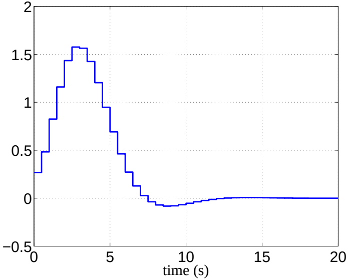
MATLAB

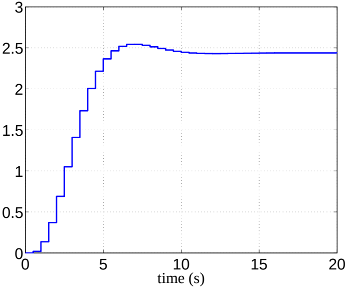
```
K=1; beta=.3; alpha=1;
G=tf(K,[1 beta alpha 0]);

ts=0.5; % sampling time
Gd=c2d(G,ts);
sysd=ss(Gd);
[A, B, C, D]=ssdata(sysd);

% Controller
polesK=[-1,-0.5+0.6*j,-0.5-0.6*j];
polesKd=exp(ts*polesK);
K=-place(A,B,polesKd);

% Observer
polesL=[-10, -9, -8];
polesLd=exp(ts*polesL);
L=place(A',C',polesLd)';
```

MATLAB

```
% Closed-loop system, state=[x;xhat]

bigA=[A,B*K;L*C,A+B*K-L*C];
bigB=[B;B];
bigC=[C,zeros(1,3)];
bigD=0;
clsys=ss(bigA,bigB,bigC,bigD,ts);

x0=[1 1 1]';    % Initial state
xhat0=[0 0 0]'; % Initial estimate
T=20;
initial(clsys, [x0;xhat0],T);
pause

t=(0:ts:T)';
v=ones(size(t));
lsim(clsys,v);
```

EXAMPLE: CONTROL OF A DC MOTOR





$$x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v(k) \equiv 0$$

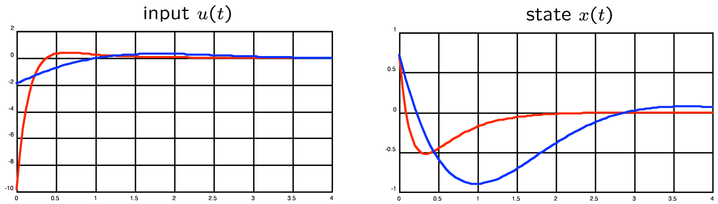
$$x(0) = \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v(k) \equiv 1$$

LINEAR QUADRATIC REGULATION

LINEAR QUADRATIC REGULATION (LQR)

- State-feedback control via pole placement requires one to assign the closed-loop poles
- Any way to place closed-loop poles automatically and optimally ?
- The main control objectives are
  1. Make the state $x(k)$ "small" (to converge to the origin)
  2. Use "small" input signals $u(k)$ (to minimize actuators' effort)

These are conflicting goals !



- LQR is a technique to place automatically and optimally the closed-loop poles

FINITE-TIME OPTIMAL CONTROL

- Consider the linear system $x(k+1) = Ax(k) + Bu(k)$ with initial condition $x(0)$
- We look for the optimal sequence of inputs

$$U = \{u(0),\ u(1),\ \ldots,\ u(N-1)\}$$

driving the state $x(k)$ towards the origin and that minimizes the performance index

$$J(x(0), U) = x'(N)Q_N x(N) + \sum_{k=0}^{N-1} x'(k)Qx(k) + u'(k)Ru(k)$$ quadratic cost

where $Q = Q' \succeq 0$, $R = R' \succ 0$, $Q_N = Q'_N \succeq 0$[6]

[6] For a matrix $Q \in \mathbb{R}^{n\times n}$, $Q \succ 0$ means that Q is a **positive definite** matrix, i.e., $x'Qx > 0$ for all $x \neq 0$, $x \in \mathbb{R}^n$. $Q_N \succeq 0$ means **positive semidefinite**, $x'Qx \geq 0, \forall x \in \mathbb{R}^n$

FINITE-TIME OPTIMAL CONTROL

- Example: Q diagonal $Q = \mathrm{diag}(q_1, \ldots, q_n)$, single input, $Q_N = 0$

$$J(x(0), U) = \sum_{k=0}^{N-1} \left(\sum_{i=1}^{n} q_i x_i^2(k) \right) + Ru^2(k)$$

- Consider again the general **linear quadratic (LQ)** problem

$$J(x(0), U) = x'(N)Q_N x(N) + \sum_{k=0}^{N-1} x'(k)Qx(k) + u'(k)Ru(k)$$

  - N is called the time horizon over which we optimize performance
  - The first term $x'Qx$ penalizes the deviation of x from the desired target $x = 0$
  - The second term $u'Ru$ penalizes actuator authority
  - The third term $x'(N)Q_N x(N)$ penalizes how much the final state $x(N)$ deviates from the target $x = 0$
- Q, R, Q_N are the tuning parameters of optimal control design (cf. the parameters of the PID controller K_p, T_i, T_d)

MINIMUM-ENERGY CONTROLLABILITY

- Consider again the problem of controllability of the state to zero with minimum energy input

$$\begin{array}{ll} \min_U & \left\| \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \right\| \\ \text{s.t.} & x(N) = 0 \end{array}$$

- The minimum-energy control problem can be seen as a particular case of the LQ optimal control problem by setting

$$R = I,\ Q = 0,\ Q_N = \infty \cdot I$$

SOLUTION TO LQ OPTIMAL CONTROL PROBLEM

- By substituting $x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^i Bu(k-1-i)$ in

$$J(x(0),U) = \sum_{k=0}^{N-1} x'(k)Qx(k) + u'(k)Ru(k) + x'(N)Q_N x(N)$$

we obtain

$$J(x(0),U) = \frac{1}{2}U'HU + x(0)'FU + \frac{1}{2}x(0)'Yx(0)$$

where $H = H' \succ 0$ is a positive definite matrix

- The optimizer U^* is obtained by zeroing the gradient

$$\begin{array}{rcl} 0 &=& \nabla_U J(x(0),U) = HU + F'x(0) \\ \longrightarrow\ U^* &=& \left[\begin{array}{c} u^*(0) \\ u^*(1) \\ \vdots \\ u^*(N-1) \end{array}\right] = -H^{-1}F'x(0) \end{array}$$

[LQ PROBLEM MATRIX COMPUTATION]

$$J(x(0),U) = x'(0)Qx(0) + \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N-1) \\ x(N) \end{bmatrix}' \underbrace{\begin{bmatrix} Q & 0 & 0 & \dots & 0 \\ 0 & Q & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & Q & 0 \\ 0 & 0 & \dots & 0 & Q_N \end{bmatrix}}_{\bar{Q}} \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N-1) \\ x(N) \end{bmatrix} +$$

$$\begin{bmatrix} u'(0) & u'(1) & \dots & u'(N-1) \end{bmatrix} \underbrace{\begin{bmatrix} R & 0 & \dots & 0 \\ 0 & R & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & R \end{bmatrix}}_{\bar{R}} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}$$

$$\begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} = \overbrace{\begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}}^{\bar{S}} \begin{bmatrix} u(0) \\ u(1) \\ \dots \\ u(N-1) \end{bmatrix} + \underbrace{\begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{\bar{N}} x(0)$$

$$\begin{aligned} J(x(0),U) &= x'(0)Qx(0) + (\bar{S}U + \bar{N}x(0))'\bar{Q}(\bar{S}U + \bar{N}x(0)) + U'\bar{R}U \\ &= \frac{1}{2}U' \underbrace{2(\bar{R} + \bar{S}'\bar{Q}\bar{S})}_{H} U + x'(0) \underbrace{2\bar{N}'\bar{Q}\bar{S}}_{F} U + \frac{1}{2}x'(0) \underbrace{2(Q + \bar{N}'\bar{Q}\bar{N})}_{Y} x(0) \end{aligned}$$

SOLUTION TO LQ OPTIMAL CONTROL PROBLEM

- The solution

$$U^* = \begin{bmatrix} u^*(0) \\ u^*(1) \\ \vdots \\ u^*(N-1) \end{bmatrix} = -H^{-1}F'x(0)$$

  is an open-loop one: $u(k) = f_k(x(0)), k = 0, 1, \ldots, N-1$

- Moreover the dimensions of the H and F matrices is proportional to the time horizon N

- We use optimality principles next to find a better solution (computationally more efficient, and more elegant)

DYNAMIC PROGRAMMING

- Consider the following basic fact in optimization

$$V_0 \triangleq \min_{z,y} f(z,y) = \min_z \{ \underbrace{\min_y f(z,y)}_{\text{this is a function of } z} \}$$

- In case f is separable in the sum of two functions

$$f(z,y) \triangleq f_0(z) + f_1(z,y)$$

  we get $\min_y f(z,y) = f_0(z) + \min_y f_1(z,y)$
- Therefore we can compute V_0 in two steps

$$\begin{aligned} V_1(z) &= \min_y f_1(z,y) \\ V_0 &= \min_z \{f_0(z) + V_1(z)\} \end{aligned}$$

- We apply the above reasoning to $f = J(x(0), U)$, $z = [u'(0) \ \ldots \ u(k_1-1)']'$, $y = [u'(k_1) \ \ldots \ u(N-1)']'$

- At a generic instant k_1 and state $x(k_1) = z$ consider the optimal **cost-to-go**

$$V_{k_1}(z) = \min_{u(k_1),\ldots,u(N-1)} \left\{ \sum_{k=k_1}^{N-1} x'(k)Qx(k) + u'(k)Ru(k) + x'(N)Q_N x(N) \right\}$$

Principle of dynamic programming

$$\begin{aligned} V_0(x(0)) &= \min_{U \triangleq \{u(0),\ldots,u(N-1)\}} J(x(0), U) \\ &= \min_{u(0),\ldots,u(k_1-1)} \left\{ \sum_{k=0}^{k_1-1} x'(k)Qx(k) + u'(k)Ru(k) + V_{k_1}(x(k_1)) \right\} \end{aligned}$$

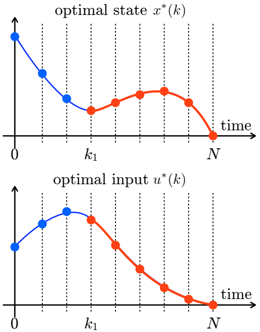
- Starting at $x(0)$, the minimum cost over $[0, N]$ equals the minimum cost spent until step k_1 plus the optimal cost-to-go from k_1 to N starting at $x(k_1)$

BELLMAN'S PRINCIPLE OF OPTIMALITY

Bellman's principle

Given the optimal sequence $U^* = [u^*(0), \ldots, u^*(N-1)]$ (and the corresponding optimal trajectory $x^*(k)$), the subsequence $[u^*(k_1), \ldots, u^*(N-1)]$ is optimal for the problem on the horizon $[k_1, N]$, starting from the optimal state $x^*(k_1)$

Richard Bellman (1920-1984)



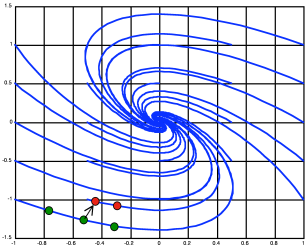
- Given the state $x^*(k_1)$, the optimal input trajectory u^* on the remaining interval $[k_1, N]$ only depends on $x^*(k_1)$
- Then each optimal move $u^*(k)$ of the optimal trajectory on $[0, N]$ only depends on $x^*(k)$
- The optimal control policy can be always expressed in state feedback form $u^*(k) = u^*(x^*(k))$!

BELLMAN'S PRINCIPLE OF OPTIMALITY

- The principle also applies to nonlinear systems and/or non-quadratic cost functions: the optimal control law can be always written in state-feedback form

$$u^*(k) = f_k(x^*(k)), \quad \forall k = 0, \ldots, N-1$$

optimal state trajectories x^*



- Compared to the open-loop solution $\{u^*(0), \ldots, u^*(N-1)\} = f(x(0))$ the feedback form $u^*(k) = f_k(x^*(k))$ has the big advantage of being more robust with respect to perturbations: at each time k we apply the best move on the remaining period $[k, N]$

RICCATI ITERATIONS

By applying the dynamic programming principle, we can compute the optimal inputs $u^*(k)$ recursively as a function of $x^*(k)$ (**Riccati iterations**):

1. Initialization: $P(N) = Q_N$
2. For $k = N, \ldots, 1$, compute recursively the following matrix

$$P(k-1) = Q - A'P(k)B(R + B'P(k)B)^{-1}B'P(k)A + A'P(k)A$$

3. Define

$$K(k) = -(R + B'P(k+1)B)^{-1}B'P(k+1)A$$

The optimal input is

$$u^*(k) = K(k)x^*(k)$$

Jacopo Francesco Riccati
(1676–1754)

The optimal input policy $u^*(k)$ is a (linear time-varying) state feedback !

- Consider the infinite-horizon optimal control problem

$$V^\infty(x(0)) = \min_{u(0),u(1),\ldots} \sum_{k=0}^{\infty} x'(k)Qx(k) + u'(k)Ru(k)$$

Result

Let (A,B) be a stabilizable pair, $R \succ 0, Q \succeq 0$. There exists a unique solution P_∞ of the **algebraic Riccati equation (ARE)**

$$P_\infty = A'P_\infty A + Q - A'P_\infty B(B'P_\infty B + R)^{-1}B'P_\infty A$$

such that the optimal cost is $V^\infty(x(0)) = x'(0)P_\infty x(0)$ and the optimal control law is the constant linear state feedback $u(k) = K_{\rm LQR}x(k)$ with

$$K_{\rm LQR} = -(R + B'P_\infty B)^{-1}B'P_\infty A.$$

MATLAB
P_∞ = dare(A,B,Q,R)

MATLAB
[-K_∞,P$_\infty$] = dlqr(A,B,Q,R)

LINEAR QUADRATIC REGULATION

- Go back to Riccati iterations: starting from $P(\infty) = P_\infty$ and going backwards we get $P(j) = P_\infty, \forall j \geq 0$
- Accordingly, we get

$$K(j) = -(R + B'P_\infty B)^{-1}B'P_\infty A \triangleq K_{\rm LQR},\ \forall j = 0, 1, \ldots$$

- The LQR control law is linear and time-invariant

MATLAB
» [-K_∞,P$_\infty$,E] = lqr(sysd,Q,R)

E= closed-loop poles

= eigenvalues of $(A + BK_{\rm LQR})$

- Closed-loop stability is ensured if (A, B) is stabilizable, $R \succ 0$, $Q \succeq 0$, and $(A, Q^{\frac{1}{2}})$ is detectable, where $Q^{\frac{1}{2}}$ is the **Cholesky factor**[7] of Q
- LQR is an automatic and optimal way of placing poles !
- A similar result holds for continuous-time linear systems (**MATLAB**: lqr)

[7] Given a matrix $Q = Q' \succeq 0$, its Cholesky factor is an upper-triangular matrix C such that $C'C = Q$ (**MATLAB**: chol)

LQR WITH OUTPUT WEIGHTING

- We often want to regulate only $y(k) = Cx(k)$ to zero, so define

$$V^\infty(x(0)) = \min_{u(0),u(1),\ldots} \sum_{k=0}^{\infty} y'(k)Q_y y(k) + u'(k)Ru(k)$$

- The problem is again an LQR problem with equivalent state weight $Q = C'Q_yC$

MATLAB

» [-K_∞,P$_\infty$,E] = dlqry(sysd,Qy,R)

Corollary

Let (A, B) stabilizable, (A, C) detectable, $R > 0$, $Q_y > 0$. The LQR control law $u(k) = K_{\rm LQR}x(k)$ the asymptotically stabilizes the closed-loop system

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} u(t) = 0$$

Intuitively: the minimum cost $x'(0)P_\infty x(0)$ is finite $\Rightarrow y(k) \to 0$ and $u(k) \to 0$.

$y(k) \to 0$ implies that the observable part of the state $\to 0$. As $u(k) \to 0$, the unobservable states remain undriven and go to zero spontaneously (=detectability condition)

LQR EXAMPLE

- Two-dimensional single input single output (SISO) dynamical system (double integrator)

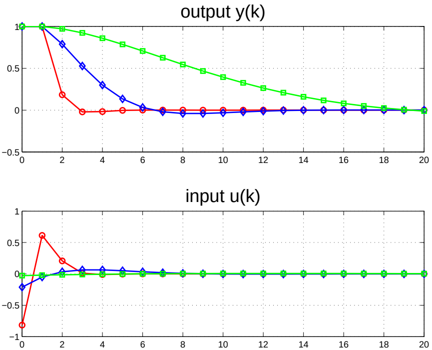
$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned}$$

- LQR (infinite horizon) controller defined on the performance index

$$V^\infty(x(0)) = \min_{u(0),u(1),\ldots} \sum_{k=0}^{\infty} \frac{1}{\rho} y^2(k) + u^2(k), \quad \rho > 0$$

- Weights: $Q = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \cdot \frac{1}{\rho} \cdot \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right] = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & 0 \end{bmatrix}, R = 1$
- Note that only the ratio $Q_{11}/R = \frac{1}{\rho}$ matters, as scaling the cost function does not change the optimal control law

LQR EXAMPLE



$\rho = 0.1$ (red line)

$$K = [-0.8166 \quad -1.7499]$$

$\rho = 10$ (blue line)

$$K = [-0.2114 \quad -0.7645]$$

$\rho = 1000$ (green line)

$$K = [-0.0279 \quad -0.2505]$$

Initial state: $x(0) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$

$$V^\infty(x(0)) = \min_{u(0),u(1),\ldots} \sum_{k=0}^{\infty} \frac{1}{\rho} y^2(k) + u^2(k)$$

KALMAN FILTERING

- **Problem:** assign observer poles in an optimal way, that is to minimize the state estimation error $\tilde{x} = x - \hat{x}$
- Information comes in two ways: from sensors measurements (a posteriori) and from the model of the system (a priori)
- We need to mix the two information sources optimally, given a probabilistic description of their reliability (sensor precision, model accuracy)

Rudolf E. Kalman*
(1930–2016)

The **Kalman filter** solves this problem, and is now the most used state observer in most engineering fields (and beyond)

* R.E. Kalman receiving the Medal of Science from the President of the USA on October 7, 2009

PROCESS MODEL

- The process is modeled as the **linear time-varying system with noise**

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + G(k)\xi(k) \\ y(k) &= C(k)x(k) + D(k)u(k) + \zeta(k) \\ x(0) &= x_0 \end{aligned}$$

- $\xi(k) \in \mathbb{R}^q$ = **process noise**. We assume $E[\xi(k)] = 0$ (zero mean), $E[\xi(k)\xi'(j)] = 0\, \forall k \neq j$ (white noise), and $E[\xi(k)\xi'(k)] = Q(k) \succeq 0$ (covariance matrix)
- $\zeta(k) \in \mathbb{R}^p$ = **measurement noise**, $E[\zeta(k)] = 0$, $E[\zeta(k)\zeta'(j)] = 0\, \forall k \neq j$, $E[\zeta(k)\zeta'(k)] = R(k) \succ 0$
- $x_0 \in \mathbb{R}^n$ is a random vector, $E[x_0] = \bar{x}_0$, $E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)'] = Var[x_0] = P_0, P_0 \succeq 0$
- Vectors $\xi(k), \zeta(k), x_0$ are uncorrelated: $E[\xi(k)\zeta'(j)] = 0$, $E[\xi(k)x_0'] = 0$, $E[\zeta(k)x_0'] = 0, \forall k, j \in \mathbb{Z}$
- Probability distributions: we often assume **normal** (=**Gaussian**) distributions $\xi(k) \sim \mathcal{N}(0, Q(k)), \zeta(k) \sim \mathcal{N}(0, R(k)), x_0 \sim \mathcal{N}(\bar{x}_0, P_0)$

Introduce some quantities:

$\hat{x}(k\|k-1)$	state estimate at time k based on data up to time $k-1$
$\tilde{x}(k\|k-1) = x(k) - \hat{x}(k\|k-1)$	state estimation error
$P(k\|k-1) = E\left[\tilde{x}(k\|k-1)\tilde{x}(k\|k-1)'\right]$	covariance of state estimation error
$\hat{x}(k\|k)$	state estimate at time k based on data up to time k
$\tilde{x}(k\|k) = x(k) - \hat{x}(k\|k)$	state estimation error
$P(k\|k) = E\left[\tilde{x}(k\|k)\tilde{x}(k\|k)'\right]$	covariance of state estimation error
$\hat{x}(k+1\|k)$	state prediction at time $k+1$ based on data up to time k

KALMAN FILTER

- The Kalman filter provides the optimal estimate $\hat{x}(k|k)$ of $x(k)$ given the measurements up to time k

- Optimality means that the trace of the variance $P(k+1|k)$ is minimized

- The filter is based on two steps:

1. measurement update based on the most recent $y(k)$

$$\begin{aligned} M(k) &= P(k|k-1)C(k)'[C(k)P(k|k-1)C(k)' + R(k)]^{-1} \\ \hat{x}(k|k) &= \hat{x}(k|k-1) + M(k)\left(y(k) - C(k)\hat{x}(k|k-1) - D(k)u(k)\right) \\ P(k|k) &= (I - M(k)C(k))P(k|k-1) \end{aligned}$$

with initial conditions $\hat{x}(0|-1) = \hat{x}_0$, $P(0|-1) = P_0$

2. time update based on the model of the system

$$\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k)$$

$$P(k+1|k) = A(k)P(k|k)A(k)' + G(k)Q(k)G(k)'$$

STATIONARY KALMAN FILTER

- Assume A, C, G, Q, R are constant.
- Under suitable assumptions[8], $P(k|k-1), M(k)$ converge to the constant matrices

$$\begin{array}{rcl} P_\infty & = & AP_\infty A' + GQG' - AP_\infty C' \left[CP_\infty C' + R\right]^{-1} CP_\infty A' \\ M & = & P_\infty C'(CP_\infty C' + R)^{-1} \end{array}$$

- By setting $L = AM$ the dynamics of the prediction $\hat{x}(k|k-1)$ becomes the Luenberger observer

$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + B(k)u(k) + L(y(k) - C\hat{x}(k|k-1) - D(k)u(k))$$

with all the eigenvalues of $(A - LC)$ inside the unit circle

MATLAB
»[~,L,P_∞,M,Z]=kalman(sys,Q,R)

$$Z = E[\tilde{x}(k|k)\tilde{x}(k|k)']$$

[8] (A, C) observable, and (A, GB_q) stabilizable, where $Q = B_q B_q'$ (B_q=Cholesky factor of Q)

TUNING KALMAN FILTERS

- It is usually hard to quantify exactly the correct values of Q and R for a given process
- The diagonal terms of R are related to how noisy are output sensors
- Q is harder to relate to physical noise, it mainly relates to how rough is the (A, B) model
- After all, Q and R are the tuning knobs of the observer (similar to LQR)
- The “larger” is R with respect to Q the “slower” is the observer to converge (L, M will be small)
- On the contrary, the “smaller” is R than Q, the more precise are considered the measurments, and the “faster” observer will be to converge

EXTENDED KALMAN FILTER

- The Kalman filter can be extended to nonlinear systems

$$\begin{aligned} x(k+1) &= f(x(k), u(k), \xi(k)) \\ y(k) &= g(x(k), u(k)) + \zeta(k) \end{aligned}$$

1. Measurement update:

$$\begin{aligned} C(k) &= \frac{\partial g}{\partial x}(\hat{x}_{k|k-1}, u(k)) \\ M(k) &= P(k|k-1)C(k)'[C(k)P(k|k-1)C(k)' + R(k)]^{-1} \\ \hat{x}(k|k) &= \hat{x}(k|k-1) + M(k)\left(y(k) - g(\hat{x}(k|k-1), u(k))\right) \\ P(k|k) &= (I - M(k)C(k))P(k|k-1) \end{aligned}$$

2. Time update:

$$\hat{x}(k+1|k) = f(\hat{x}(k|k), u(k)),\ \hat{x}(0|-1) = \hat{x}_0$$

$$A(k) = \frac{\partial f}{\partial x}(\hat{x}_{k|k}, u(k), E[\xi(k)]),\ G(k) = \frac{\partial f}{\partial \xi}(\hat{x}_{k|k}, u(k), E[\xi(k)])$$

$$P(k+1|k) = A(k)P(k|k)A(k)' + G(k)Q(k)G(k)',\ P(0|-1) = P_0$$

- The EKF is in general not optimal and may even diverge, due to linearization. But is the de-facto standard in nonlinear state estimation

LQG CONTROL

- **Linear Quadratic Gaussian (LQG)** control combines an LQR control law and a stationary Kalman predictor/filter
- Consider the stochastic dynamical system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \xi(k),\ w \sim \mathcal{N}(0, Q_{KF}) \\ y(k) &= Cx(k) + \zeta(k),\ v \sim \mathcal{N}(0, R_{KF}) \end{aligned}$$

with initial condition $x(0) = x_0, x_0 \sim \mathcal{N}(\bar{x}_0, P_0), P, Q_{KF} \succeq 0, R_{KF} \succ 0$, and ζ and ξ are independent and white noise terms.

- The objective is to minimize the cost function

$$J(x(0), U) = \lim_{T\to\infty} \frac{1}{T} E\left[\sum_{k=0}^{T} x'(k)Q_{LQ}x(k) + u'(k)R_{LQ}u(k)\right]$$

when the state x is not measurable

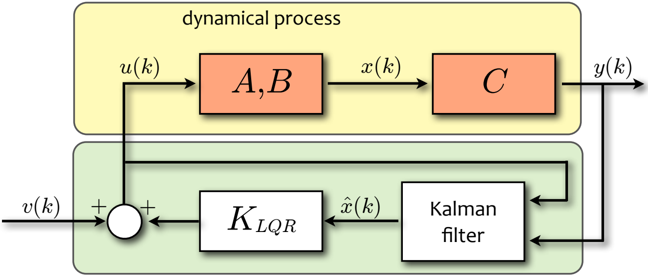
LQG CONTROL

If we assume that all the assumptions for LQR control and Kalman predictor/filter hold, i.e.

- the pair (A, B) is reachable and the pair (A, C_q) with C_q such that $Q_{LQ} = C_q C_q'$ is observable (here Q is the weight matrix of the LQ controller)
- the pair (A, B_q), with B_q s.t. $Q_{KF} = B_q B_q'$, is stabilizable, and the pair (A, C) is observable (here Q is the covariance matrix of the Kalman predictor/filter)

Then, apply the following procedure:

1. Determine the optimal stationary Kalman predictor/filter, neglecting the fact that the control variable u is generated through a closed-loop control scheme, and find the optimal gain L_{KF}
2. Determine the optimal LQR strategy assuming the state accessible, and find the optimal gain K_{LQR}



LQG controller

Analogously to the case of output feedback control using a Luenberger observer, it is possible to show that the extended state $[x' \; \tilde{x}']'$ has eigenvalues equal to the eigenvalues of $(A + BK_{LQR})$ plus those of $(A - L_{KF}C)$ ($2n$ in total)