Risk measures

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Outline

1. Mathematical preliminaries
2. Risk measures
3. Examples
4. Law invariance

I. Mathematical preliminaries

- ✓ Dual topological spaces
- ✓ The w and w$*$ topologies
- ✓ $\mathcal{L}_p$ spaces

Dual topological space

Let $\mathcal{Z}$ be a **Banach** space[1] The **algebraic dual** of $\mathcal{Z}$ is the space

$$\mathcal{Z}^{\#} := \{f : \mathcal{Z} \to \mathbb{R} : \text{linear}\}$$

The (topological) **dual** of $\mathcal{Z}$ is the space

$$\mathcal{Z}^{*} = \{f \in \mathcal{Z}^{\#} : \text{continuous}\}$$

[1] i.e., a normed space $(\mathcal{Z}, \|\cdot\|)$ which is **complete** with respect to the metric $\rho(x, y) = \|x - y\|$.

The weak topology

Let $\mathcal{Z}$ be a Banach space and $\mathcal{Z}^*$ its dual. The **weak topology** on $\mathcal{Z}$ is the weakest (smallest) topology that makes the elements of $\mathcal{Z}^*$ continuous. Its basic sets are

$$W_{(x,f_1,\ldots,f_s,\epsilon)} = \{z \in \mathcal{Z} : |f_i(z) - f_i(x)| < \epsilon, \forall i \in \mathbb{N}_{[1,s]}\}.$$

The weak topology

A sequence $\{x_n\}_n \subseteq \mathcal{Z}$ **converges weakly** to an element $x \in \mathcal{Z}$, denoted by

$$x_n \xrightarrow{w} x,$$

if for every finite selection of elements of $\mathcal{Z}^*$, $f_1, \ldots, f_s$ it is

$$|f_i(x_n) - f_i(x)| \to 0.$$

The weak$*$ topology

The **weak$*$ topology** on $\mathcal{Z}^*$ is the weakest (smallest) topology which makes the maps $f \mapsto f(x)$ continuous. Its basic sets are

$$W^*_{(f,x_1,\ldots,x_s,\epsilon)} = \{g \in \mathcal{Z}^* : |g(x_i) - f(x_i)| < \epsilon, \forall i \in \mathbb{N}_{[1,s]}\}.$$

The weak$*$ topology

Let $\mathcal{Z}$ be a Banach space and $\mathcal{Z}^*$ its dual. A sequence $\{f_n\}_n \subseteq \mathcal{Z}^*$ **converges weakly**$*$ to an element $f \in \mathcal{Z}^*$, denoted by

$$f_n \xrightarrow{w*} f,$$

if for every $x_1, \ldots, x_s \in \mathcal{Z}$, it is

$$|f_n(x_i) - f(x_i)| \to 0.$$

This is exactly the notion of **pointwise convergence**, i.e., $f_n \xrightarrow{w*} f$ means that the sequence of function f_n converges pointwise.

$\mathcal{L}_p$ spaces

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. The space $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ – $p \in [1, \infty)$ – is a space of random variables

$$Z : (\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$$

($\mathcal{F}$-measurable) equipped with the p-norm:

$$\|Z\|_p = \left(\int_\Omega |Z|^p \mathrm{dP} \right)^{1/p} = \mathbb{E}[|Z|^p]^{1/p}$$

so that the above integral is **well-defined** and **finite**.

$\mathcal{L}_p$ spaces

If $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ with $p \in [1, \infty)$, then the p-th order of Z is **finite** and all its orders $1 \leq p' \leq p$ are also **finite**[2], so

$$\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \subseteq \mathcal{L}_{p'}(\Omega, \mathcal{F}, \mathrm{P})$$

[2]W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill Book Co., New York, 1987, Chap. 3, Ex. 7

The space $\mathcal{L}_\infty$

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. The space $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ is a space of random variables

$$Z : (\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$$

($\mathcal{F}$-measurable) equipped with the ∞-norm:

$$\|Z\|_\infty = \operatorname*{ess\,sup}_{\omega \in \Omega} |Z(\omega)|,$$

so that $\|Z\|_\infty$ is **finite**.

(In)equality on $\mathcal{L}_p$ spaces

Let Z and V be two random variables of $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$. We consider those **equal** if

$$\mathrm{P}(\{\omega \in \Omega : Z(\omega) \neq V(\omega)\}) = 0,$$

or, what is the same

$$\mathrm{P}(Z \neq V) = 0,$$

and we denote $Z = V$ or $Z \equiv V$[3]. Similarly, $Z \leq V$ means

$$\mathrm{P}(Z > V) = 0.$$

The relation $\leq$ is a **partial order** on $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$.

[3] or, sometimes, $Z = V$ a.s. We also define the space $L_p(\Omega, \mathcal{F}, \mathrm{P}) = \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) / \equiv$

Topological dual of $\mathcal{L}_p$ spaces

Let X be a Banach space. The **topological dual** of X, denoted X^*, is the space of linear continuous (bounded) functional from X to $\mathbb{R}$. For $x^* \in X^*$ we define the scalar product between x^* and x as

$$\langle x^*, x\rangle = x^*(x)$$

For $\mathcal{L}_p$-spaces with $p \in (1, \infty)$we have

$$(\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}))^* = \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P}),$$

where $q^{-1} + p^{-1} = 1$ and

$$(\mathcal{L}_1(\Omega, \mathcal{F}, \mathrm{P}))^* = \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P}).$$

The dual of $\mathcal{L}_\infty$ is not an $\mathcal{L}_p$-space.

Topological dual of $\mathcal{L}_p$ spaces

An element $Y \in (\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}))^*$ is a continuous linear operator

$$Y : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \ni Z \mapsto Y(Z) \in \mathbb{R}$$

where

$$Y(Z) := \langle Y, Z \rangle = \int_\Omega \tilde{Y} X \mathrm{d}P$$

where $\tilde{Y} \in \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P})$. We simply identify Y with $\tilde{Y}$[4].

[4]Indeed, such a $\tilde{Y}$ exists; see Theorem 4.4.10 in: D.S. Bridges, *Foundations of real and abstract analysis*, Springer, 1998, p. 202

Dual of $\mathcal{L}_p$

The dual of $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ — for $p \in [1, \infty)$ — is the space $\mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P})$, where $q^{-1} + p^{-1} = 1$. For $u \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ and $v \in \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P})$ we define their scalar-valued product as

$$\langle u, v \rangle = \int_\Omega uv \mathrm{d}P.$$

II. Risk measures

- ✓ Risk measures
- ✓ Coherent risk measures
- ✓ Characterisation of coherency

Motivation

Stochastic shortest path

1. Find the shortest path on a directed graph which connects nodes A and B
2. The cost of moving from node i to node j is a normally distributed random variable X_{ij} with known mean μ_{ij} and variance σ_{ij}
3. Costs X_{ij} are independent random variables
4. We are not allowed to change our decision on the way (X_{ij} are not measured)

Motivation

The **stochastic** shortest path problem amounts to the minimisation of

$$\mathbb{E}[\sum_i X_{i,i^+}] = \sum_i \mu_{i,i^+},$$

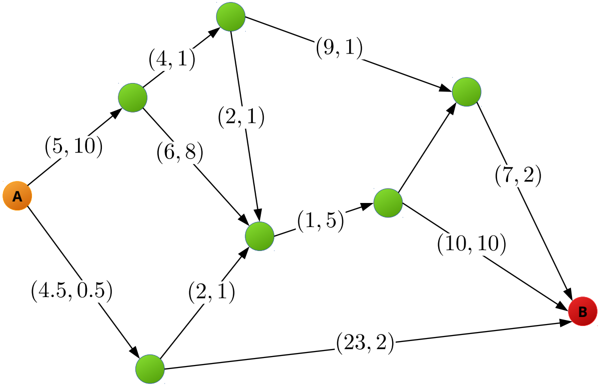
totally disregards any stochastic information (e.g., the variance). Let us define

$$J = \mathbb{E}[X] + \lambda \operatorname{Var}[X],$$

where X is the total cost. This is known as the **Markowitz** risk measure[5].

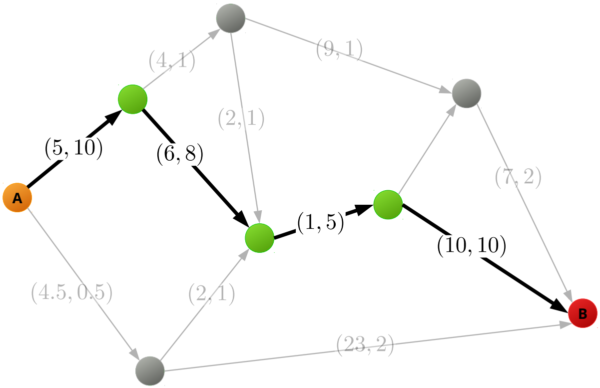
[5] Markowitz's mean-variance risk measure was one of the first ways to quantify risk in 1952. However, nowadays it doesn't qualify as a "good" risk measure for reasons we'll explain in a while.

Motivation



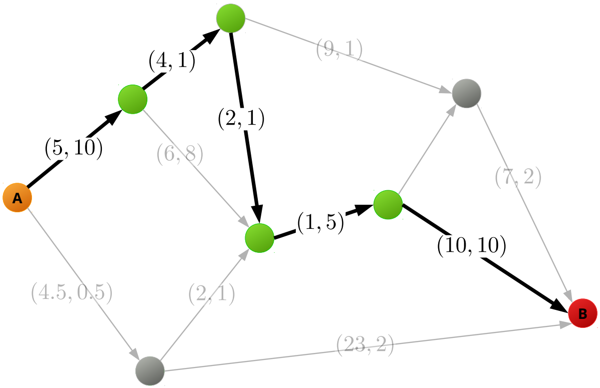
$(\mathbb{E}[x_{i,j}], \mathrm{Var}[x_{ij}])$

Motivation



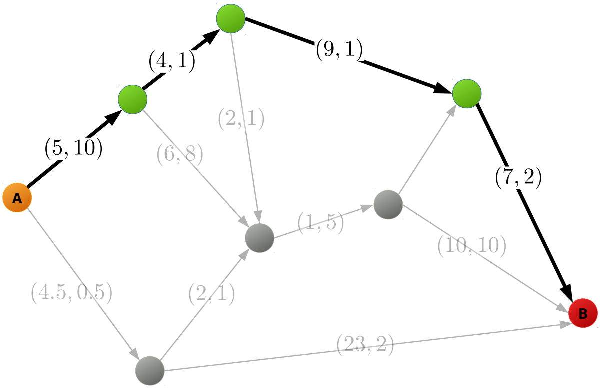
$\mathbb{E}[X] = 22, \mathrm{Var}X = 33, J = 352, \mathrm{P}[X > 30] = 8.18\%$

Motivation



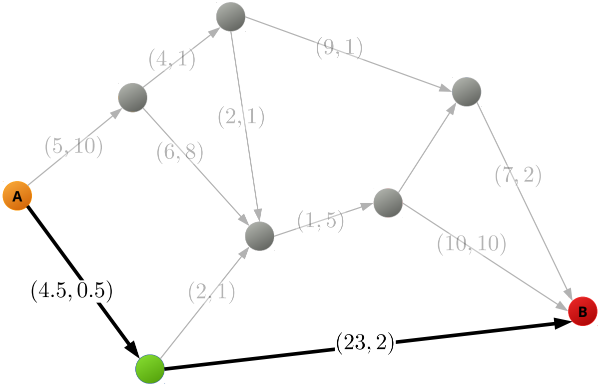
$$\mathbb{E}[X] = 22, \mathrm{Var} X = 27, J = 292, \mathrm{P}[X > 30] = 6.18\%$$

Motivation



$\mathbb{E}[X] = 25, \mathrm{Var}X = 14, J = 165, \mathrm{P}[X > 30] = 9.1\%$

Motivation



$\mathbb{E}[X] = 27, \mathrm{Var}X = 2.5, J = 52, \mathrm{P}[X > 30] = 5.69\%$

Motivation

Notice that:

- The stochastic shortest path solution with $\mathbb{E}[X] = 22$ is too risky since $\mathrm{P}[X > 30] = 8.18\%$ and $\mathrm{Var}[X] = 33$
- The minimum mean-variance solution with $\mathbb{E}[X] = 27$ is a much wiser choice since $\mathrm{P}[X > 30] = 5.69\%$ and $\mathrm{Var}[X] = 2.5$
- The stochastic solution has a mean-risk of $J = 352$, whereas the mean-risk-optimum is at $J = 52$

Conclusion: $\mathbb{E}[\cdot]$ may often be a bad idea...

Risk measures

A **risk measure** is a function $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$.

Risk measures

A **risk measure** is a function $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$.

Assumption: ρ is **proper**, i.e., for all $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$, $\rho(Z) > -\infty$ and $\operatorname{dom} \rho = \{Z : \rho(Z) < \infty\} \neq \varnothing$.

Risk measures

Since risk measures are defined on $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ it is implied that for $Z_1, Z_2 \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$

$$Z_1 = Z_2 \Rightarrow \rho(Z_1) = \rho(Z_2),$$

and recall that $Z_1 = Z_2$ means that

$$\mathrm{P}[Z_1 \neq Z_2] = 0$$

Coherent risk measures

Let $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$. A risk measure is called **coherent** if it satisfies the following assumptions for $Z, V \in \mathcal{Z}$, $\lambda \geq 0$ and $a \in \mathbb{R}$

1. Subadditivity

$$\rho(Z + V) \leq \rho(Z) + \rho(V)$$

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2. Positive homog.
$$\rho(\lambda Z) \leq \lambda \rho(Z)$$

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$$\rho(Z+V) \leq \rho(Z) + \rho(V)$$
2. Positive homog.
$$\rho(\lambda Z) \leq \lambda \rho(Z)$$
3. Monotonicity
$$Z \leq V \Rightarrow \rho(Z) \leq \rho(V)$$
4. Translation invariance
$$\rho(Z+a) = a + \rho(Z)$$

Coherent risk measures are convex

Because of the sub-additivity and Positive homogeneity of coherent risk measures, they are **convex**

$$\rho(\lambda Z + (1-\lambda)V) \leq \lambda\rho(Z) + (1-\lambda)\rho(V),$$

for all $Z, V \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ and $\lambda \in [0,1]$.

* Bibliographic note

The four coherence conditions are the ones originally stipulated by Artzner *et al.*, 1997.

Shapiro, Dentcheva and Ruszczyński use instead: (i) convexity, (ii) monotonicity, (iii) translation equivariance and (iv) positive homogeneity. These two conventions are of course equivalent.

Convex conjugate mappings

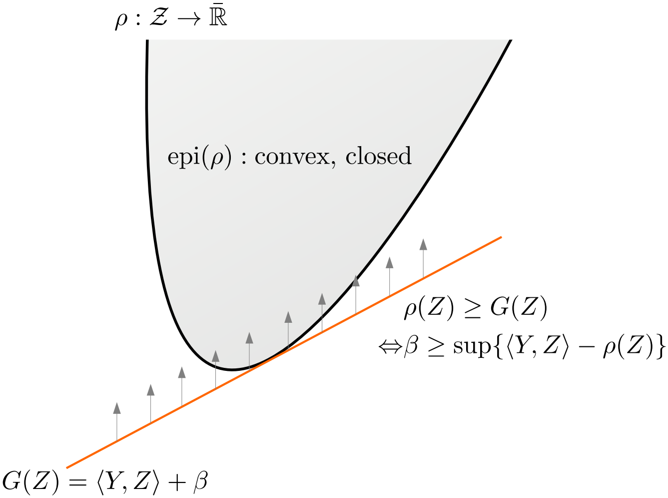
A convex mapping $\rho : \mathcal{Z} \to \bar{R}$ can be seen

- As its graph described by $Z \mapsto \rho(Z)$
- As its epigraph, i.e., $\mathrm{epi}(\rho) = \{(Z, \alpha) : \rho(Z) \leq \alpha\}$
- And, when the epigraph is closed, it can be written as the intersection of halfspaces defined by its **tangents**, i.e., affine functions

$$G(Z) = \langle Y, Z \rangle + \beta,$$

with $\rho(Z) \geq G(Z)$ for all Z.

Convex conjugate mappings



Conjugate risk measures

For a coherent risk measure $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$, the **conjugate** risk measure is a function $\rho^* : \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$ defined as

$$\rho^*(Y) = \sup_{Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})} \{\langle Z, Y \rangle - \rho(Z)\},$$

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and the **bi-conjudate** of ρ is a function $\rho^{**} : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$ with

$$\rho^{**}(Z) = \sup_{Y \in \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P})} \{\langle Y, Z \rangle - \rho^*(Y)\},$$

Conjugate risk measures

For a coherent risk measure $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathrm{I\!R}}$, the **conjugate** risk measure is a function $\rho^* : \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathrm{I\!R}}$ defined as

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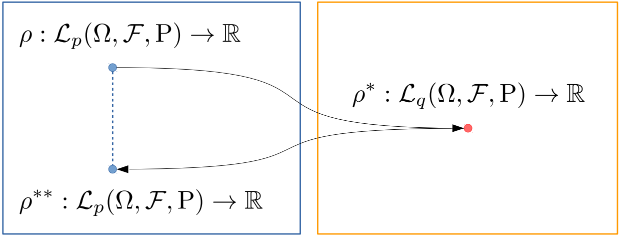
If ρ is lower semicontinuous, then, by the **Fenchel-Moreau** theorem,

$$\rho = \rho^{**}.$$

Conjugate risk measures

$$(\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}))^* = \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P})$$



Convex conjugate mappings

Convex conjugates possess many interesting properties

1. Fenchel's inequality

$$\langle Z, Y \rangle \leq \rho(Z) + \rho(Y),$$

2. If ρ is proper, convex, closed, then $\inf \rho(Z) = -\rho^*(0)$
3. They are available in closed form for all popular convex risk measures
4. The convex conjugate of $\delta(Z \mid C)$ is the so-called *support function* of C given by

$$\delta^*(Y \mid C) = \sup_{Z \in C} \langle Y, Z \rangle.$$

The Fenchel-Moreau Theorem

Theorem. Let $\mathcal{Z}$ be a Banach space (e.g., any of $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ with $p \in [1, \infty)$) and $f : \mathcal{Z} \to \bar{\mathbb{R}}$ be a proper function. Then

$$f^{**} = \mathrm{cl} f.$$

If, additionally, f is lsc, then

$$f^{**} = f.$$

The Fenchel-Moreau Theorem

Corollary. Assume that $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$ is a proper, convex, lower semi-continuous risk measure. Then,

$$\begin{aligned}\rho(Z) &= \rho^{**}(Z) \\ &= \sup_{Y \in \mathcal{L}_q(\Omega,\mathcal{F},\mathrm{P})} \left\{ \langle Y, Z \rangle - \rho^*(Y) \right\}, \\ &= \sup_{Y \in \operatorname{dom} \rho^*} \left\{ \langle Y, Z \rangle - \rho^*(Y) \right\}.\end{aligned}$$

Monotonicity of ρ

The **monotonicity** property of a proper, convex, lsc risk measure ρ, that is

$$\rho(Z) \leq \rho(V),$$

whenever $Z \leq V$, $Z, V \in \mathcal{Z}$, holds iff the elements of the set $\operatorname{dom} \rho^*$ are **nonnegative**.

Translation equivariance of ρ

The **translation equivariance** property of a proper, convex, lsc risk measure ρ, that is

$$\rho(Z + c) = \rho(Z) + c,$$

for all $Z \in \mathcal{Z}$, holds iff every $\zeta \in \operatorname{dom} \rho^*$ satisfies

$$\int_\Omega \zeta \mathrm{dP} = 1.$$

Positive homogeneity of ρ

A proper, convex, lsc risk measure ρ is **positive homogeneous**[6], that is

$$\rho(\alpha Z) = \alpha \rho(Z),$$

for all $Z \in \mathcal{Z}$ and $\alpha \geq 0$, iff it is the **support function** of $\operatorname{dom} \rho^*$, i.e.,

$$\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \langle \zeta, Z \rangle,$$

where $\mathfrak{A} := \operatorname{dom} \rho^*$.

[6] Theorem 13.2 in R.T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, 1972.

To summarise...

Let $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathrm{I\!R}}$, $p \in [1, \infty)$, be

1. proper, convex, lsc
2. monotone
3. translation invariant

Then,

$$\rho(Z) = \sup_{Y \in \mathfrak{A}} \left\{ \langle Y, Z \rangle - \rho^*(Y) \right\},$$

where notice that $\mathfrak{A}$ is a (w$*$-closed) subset of

$$\mathfrak{P} = \left\{ Y \in \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P}), \int_\Omega Y \mathrm{dP} = 1, Y \geq 0 \right\}$$

To summarise...

Let $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathrm{I\!R}}$, $p \in [1, \infty)$, be

1. proper, convex, lsc
2. monotone
3. translation invariant
4. positively homogeneous

Then,

$$\rho(Z) = \sup_{Y \in \mathfrak{A}} \langle Y, Z \rangle.$$

Characterisation of coherent risk measures

Theorem. A risk measure is **coherent** iff it is the support function of a **w$*$ closed** subset $\mathfrak{A} \subseteq \mathfrak{P}$, that is

$$\rho(Z) = \sup_{Y \in \mathfrak{A}} \langle Y, Z \rangle.$$

Subdifferentials of risk measures

Let ρ be a convex proper lower semicontinuous risk measure on $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$. Then,

$$\mathfrak{A} = \partial\rho(0),$$

and

$$\partial\rho(Z) = \arg\max_{Y \in \mathfrak{A}} \langle Y, Z \rangle$$

End of section

In summary:

- Risk measures extract a characteristic value $\rho[Z]$ out of a random variable $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$
- Risk measures are *sufficiently regular* if they satisfy four *coherence* axioms: convexity, monotonicity, translation equivariance and positive homogeneity
- Using the Fenchel-Moreau Theorem coherent risk measures can be written as the supremum of the expectation of Z, $\mathbb{E}_\mu[Z]$ over a set of a set of distributions $\mu \in \mathfrak{A}$
- Sub-differentiability and continuity properties of coherent risk measures are rather well understood

III. Examples of risk measures

- ✓ Incoherent risk measures
  - ⋆ Mean-variance
  - ⋆ Value-at-risk
- ✓ Coherent risk measures
  - ⋆ Mean-upper-semideviation of order p
  - ⋆ Average value-at-risk

Mean-variance

A simple (non coherent) risk measure is the mean-variance defined as

$$\rho(Z) = \mathbb{E}[Z] + c\mathrm{Var}[Z].$$

Value at risk

The **value at risk**[7] of a random variable $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ is defined as

$$\begin{aligned} \mathrm{V@R}_\alpha[Z] &= Q(1-\alpha) \\ &= \inf\{z \in \mathbb{R} : \mathrm{P}[Z \le z] \ge 1-\alpha\} \\ &= \inf\{z \in \mathbb{R} : \mathrm{P}[Z > z] \le \alpha\}. \end{aligned}$$

Notice that

$$\mathrm{P}[Z > \mathrm{V@R}_\alpha[Z]] = 1 - \mathrm{P}[Z \le \mathrm{V@R}_\alpha[Z]] = \alpha.$$

If $\mathrm{P}[Z = \mathrm{V@R}_\alpha[Z]] = 0$, then

$$\mathrm{P}[Z \ge \mathrm{V@R}_\alpha[Z]] = \alpha.$$

[7] $\mathrm{V@R}_\alpha$ was introduced in the '80s by *JP Morgan* for internal use and was later proposed by *Basel II* as a quantifier of market risk.

Value at risk

The value-at-risk is **not** a coherent risk measure, although it satisfies

$$\mathrm{V@R}_\alpha[Z + c] = \mathrm{V@R}_\alpha[Z] + c,$$

because it is **not subadditive** (although it turns out to be subadditive for normally distributed RVs).

Mean-upper-semideviation of order p

Let $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$, $c \in [0,1]$ and

$$\rho(Z) := \mathbb{E}[Z] + c\mathbb{E}\left[[Z - \mathbb{E}\,[Z]]_+^p\right]^{1/p},$$

Mean-upper-semideviation of order p

Let $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$, $c \in [0,1]$ and

$$\rho(Z) := \mathbb{E}[Z] + c\mathbb{E}\left[[Z - \mathbb{E}[Z]]_+^p\right]^{1/p},$$

This is clearly subadditive, translation equivariant, pos. homogeneous and

$$\mathfrak{A} = \left\{ Y \in \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P}) \;\middle|\; \begin{array}{l} Y = 1 + G - \mathbb{E}[G], \\ \|G\|_q \leq c, G \geq 0 \end{array} \right\}.$$

All elements of $\mathfrak{A}$ are positive, thus ρ is monotonous.

Mean-upper-semideviation of order $p = 1$

Assuming that $z = (z_1, \ldots, z_s)$ is discrete and $p_i = \mathrm{P}[Z = Z_i]$ we have

$$\rho[Z] = \max_{\mu,g} \mu' z$$

s.t.

$$\mu_i - p_i g_i + p_i d = p_i$$

$$d - p' g = 0$$

$$0 \leq g_i \leq c$$

Computation of MUS of order $p = 1$

```
function M = mus(Z, p, c)
% Computation using the definition
% Z  Discrete values of random variable Z
% p  Probability values
% c  Parameter c of MUS
% M  Mean—upper semideviation with parameter c

EZ = sum(p.*Z);
M = EZ + c* sum(p.*max(Z-EZ, zeros(size(Z))));
```

Computation of MUS of order $p = 1$

```
function [M, mu] = mus(Z, p, c)
% Computation using the dual representation
% M   Mean-upper semideviation
% mu  A subgradient of MUS

p=p(:); Z=Z(:);
n = length(Z);
A = [eye(n) -diag(p) p; zeros(1,n) -p' 1];
b = [p; 0];
H = [zeros(n,n) eye(n) zeros(n,1);
     zeros(n,n) -eye(n) zeros(n,1)];
K = [ kron(c,ones(n,1)); zeros(n,1)];
[u, M, flag] = linprog(-[Z(:)' zeros(1, n+1)], ...
    H, K, A,b);
assert(flag==1, 'Numerical problems');
M= -M; mu = u(1:n);
```

Average value-at-risk

The **average value-at-risk**[8] of a $Z \in \mathcal{Z} = \mathcal{L}_1(\Omega, \mathcal{F}, \mathrm{P})$ is

$$\mathrm{AV@R}_\alpha(Z) := \inf_{t \in \mathbb{R}} \{t + \alpha^{-1}\mathbb{E}[Z - t]_+\},$$

where is the $\inf$ attained?

[8] aka *expected shortfall* and *expected tail loss*.

Average value-at-risk

For given $Z \in \mathcal{Z}$ define

$$\phi(t) := t + \alpha^{-1}\mathbb{E}[Z - t]_+$$

If H_Z is continuous at t, then ϕ is differentiable at t and[9]

$$\phi'(t) = 1 + \alpha^{-1}(H_Z(t) - 1).$$

The minimum of ϕ is attained where $\phi'(t^*) = 0$, i.e., over the interval $[t^*, t^{**}]$ where

$$t^* = \inf\{t : H_Z(t) \geq 1 - \alpha\}$$
$$t^{**} = \sup\{t : H_Z(t) \leq 1 - \alpha\}$$

[9]Exercise: Prove that when $\mathbb{E}[Z - t]_+$ is diff/ble at t, then $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[Z - t]_+ = H_Z(t) - 1$. Use the fact that $\mathbb{E}[Z - t]_+ = \mathbb{E}[(Z - t)1_{[Z \geq t]}]$.

Average value-at-risk

The infimum is attained for $t \in [t^*, t^{**}]$ where

$$t^* = \inf\{t : H_Z(t) \geq 1 - \alpha\} = \mathrm{V@R}_\alpha(Z),$$

that is

$$\mathrm{AV@R}_\alpha(Z) = t^* + \alpha^{-1}\mathbb{E}[Z - t^*]_+.$$

Average value-at-risk

An **alternative representation**:

$$
\begin{aligned}
\mathrm{AV@R}_\alpha[Z] &= t^* + \alpha^{-1}\mathbb{E}[Z - t^*]_+ \\
&= t^* + \alpha^{-1}\int_{-\infty}^{+\infty}[Z - t^*]_+ \mathrm{dP} \\
&= t^* + \alpha^{-1}\int_{t^*}^{+\infty}(Z - t^*)\mathrm{dP} \\
&= t^* + \alpha^{-1}\int_{t^*}^{+\infty} Z\mathrm{dP} - t^*\alpha^{-1}\underbrace{\int_{t^*}^{+\infty}\mathrm{dP}}_{\mathrm{P}[Z \geq t^*] = \alpha} \\
&= \alpha^{-1}\int_{t^*}^{+\infty} Z\mathrm{dP}.
\end{aligned}
$$

Average value-at-risk

Assuming $\mathrm{P}[Z = \mathrm{V@R}_\alpha[Z]] = 0$ we have

$$\mathrm{P}[Z \geq \mathrm{V@R}_\alpha[Z]] = \alpha$$

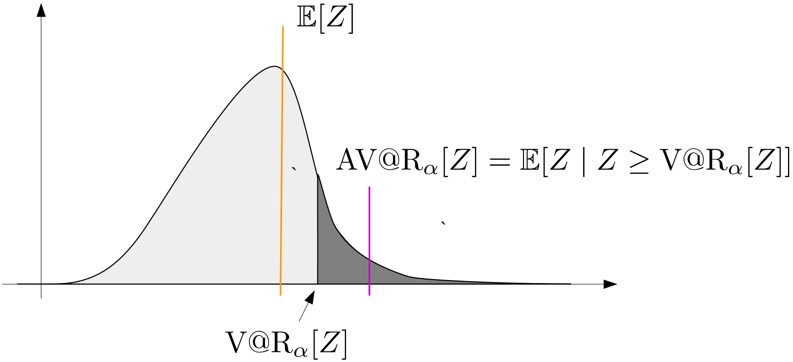
we have **another representation** of $\mathrm{AV@R}_\alpha$

$$\begin{aligned}\mathrm{AV@R}_\alpha[Z] &= \alpha^{-1} \int_{\mathrm{V@R}_\alpha[Z]}^{+\infty} Z \mathrm{dP}. \\ &= \mathbb{E}\left[Z \mid Z \geq \mathrm{V@R}_\alpha\left[Z\right]\right].\end{aligned}$$

This is because for $E \in \mathcal{F}$,

$$\mathbb{E}[Z \mid E] = \frac{\mathbb{E}[1_E Z]}{\mathrm{P}[E]}$$

Average value-at-risk



Average value-at-risk

$\mathrm{AV@R}_\alpha$ is **convex**; Indeed, for $Z \in \mathcal{Z}$ and $t \in \mathbb{R}$ the function

$$\phi(t, Z) := t + \alpha^{-1}\mathbb{E}[Z - t]_+,$$

is convex in (t, Z) and $\mathrm{AV@R}_\alpha$ is the inf-projection of ϕ — recall that

$$\mathrm{AV@R}_\alpha(Z) := \inf_{t\in\mathbb{R}}\{t + \alpha^{-1}\mathbb{E}[Z - t]_+\},$$

therefore, $\mathrm{AV@R}_\alpha$ is convex[10].

[10] See Proposition 2.22 in: R.T. Rockafellar, R. J.-B. Wets, *Variational Analysis*, Springer, Berlin 2009.

Average value-at-risk

$\mathrm{AV@R}_\alpha$ is

- ✓ monotone,
- ✓ translation equivariant,
- ✓ positively homogeneous,

thus, it is a **coherent** risk measure[11].

[11] As an exercise, verify that $\mathrm{AV@R}_\alpha$ is monotonous and translation equivariant.

Average value-at-risk

The **conjugate** function of $\mathrm{AV@R}_\alpha$ is

$$\begin{aligned}
(\mathrm{AV@R}_\alpha)^*[Y] &= \sup_{Z\in\mathcal{Z}}\{\langle Y, Z\rangle - \mathrm{AV@R}_\alpha[Z]\}\\
&= \sup_{Z\in\mathcal{Z}}\{\langle Y, Z\rangle - \inf_t\{t + \alpha^{-1}\mathbb{E}[Z-t]_+\}\}\\
&= \sup_{Z,t}\{\langle Y, Z\rangle - t - \alpha^{-1}\mathbb{E}[Z-t]_+\}\\
&= \sup_{S,t}\{\langle Y, S\rangle - \langle Y-1, t\rangle - \alpha^{-1}\mathbb{E}[S]_+\},
\end{aligned}$$

whose domain is[12]

$$\mathfrak{A} = \{Y \in \mathcal{Z}^* : \mathbb{E}[Y] = 1, Y(\omega) \in [0, \alpha^{-1}],\ \text{for a.e. } \omega \in \Omega\}.$$

[12] Exercise: Verify that this is indeed the domain of $(\mathrm{AV@R}_\alpha)^*$.

Average value-at-risk

$\mathrm{AV@R}_\alpha[Z]$ can be written as

$$\mathrm{AV@R}_\alpha[Z] = \sup_{Y \in \mathcal{Z}} \left\{ \langle Y, Z \rangle : \mathbb{E}[Y] = 1, Y \in [0, \alpha^{-1}], \text{a.e.} \right\}$$

In case $\Omega = \{\omega_1, \ldots, \omega_K\}$ and $p_i := \mathrm{P}[\omega = \omega_i]$ then $\mathrm{AV@R}_\alpha$ is computed via the following LP

$$\mathrm{AV@R}_\alpha[Z] = \max_{Y \in \mathbb{R}^K} \underbrace{\sum p_i Y_i Z_i}_{\langle Y, Z \rangle}$$

$$\text{s.t.} \quad \sum p_i Y_i = 1$$

$$0 \leq Y_i \leq \alpha^{-1}$$

Average value-at-risk

Equivalently, $\mathrm{AV@R}_\alpha$ can be computed by

$$\mathrm{AV@R}_\alpha[Z] = \max_{\mu \in \mathbb{R}^K} \underbrace{\sum_i \mu_i Z_i}_{\mathbb{E}_\mu[Z]}$$

$$\text{s.t.} \quad \sum \mu_i = 1$$

$$0 \le p_i^{-1} \mu_i \le \alpha^{-1}$$

Average value-at-risk

We may also compute $\mathrm{AV@R}_\alpha[Z]$ using the formula:

$$\begin{aligned}\mathrm{AV@R}_\alpha[Z] &= \min_{t\in\mathbb{R}} \left\{t + \alpha^{-1}\mathbb{E}[Z-t]_+\right\} \\ &= \min_{t\in\mathbb{R}} \left\{t + \alpha^{-1}\sum_i p_i[Z_i - t]_+\right\} \\ &= \min_{\substack{t\in\mathbb{R},\xi\in\mathbb{R}^n \\ \xi\geq 0, Z_i - t\leq\xi_i}} \left\{t + \alpha^{-1}\sum_i p_i\xi_i\right\}\end{aligned}$$

Subdifferential of $\mathrm{AV@R}_\alpha[\cdot]$

$\mathrm{AV@R}_\alpha$ is **subdifferentiable** with

$$\begin{aligned}\partial(\mathrm{AV@R}_\alpha)[Z] &= \arg\max_{Y\in\mathfrak{A}} \langle Y, Z\rangle \\ &= \arg\max_{Y\in\mathcal{Z}^*} \left\{\langle Y, Z\rangle : Y \in [0, \alpha^{-1}], \text{a.e. }, \mathbb{E}[Y] = 1\right\}.\end{aligned}$$

Relaxing the equality constraint $\mathbb{E}[Y] = 1$ we have the **Lagrangian**

$$\begin{aligned}L(Y, \lambda; Z) &= \langle Y, Z\rangle + \lambda(1 - \mathbb{E}[Y]) \\ &= \langle Y, Z\rangle + \lambda - \langle \lambda, Y\rangle \\ &= \langle Y, Z - \lambda\rangle + \lambda.\end{aligned}$$

Subdifferential of $\mathrm{AV@R}_\alpha[\cdot]$

We can now introduce the **dual** function

$$\begin{aligned} q(\lambda) &= \sup_{Y \in [0,\alpha^{-1}]} L(Y,\lambda) \\ &= \sup_{Y \in [0,\alpha^{-1}]} \langle Y, Z-\lambda \rangle + \lambda \\ &= \sup_{Y \in [0,\alpha^{-1}]} \int Y(Z-\lambda)\mathrm{dP} + \lambda, \end{aligned}$$

the supremum is attained for $Y = \alpha^{-1} 1_{[Z-\lambda \geq 0]}$, so

$$q(\lambda) = \alpha^{-1}\mathbb{E}[Z-\lambda]_+ + \lambda.$$

Subdifferential of $\mathrm{AV@R}_\alpha[\cdot]$

The **dual problem** is[13]

$$\min_{\lambda \in \mathbb{R}} \alpha^{-1}\mathbb{E}[Z-\lambda]_+ + \lambda.$$

The set of its minimizers is a bounded set, so we have strong duality.

[13] Does it ring a bell? This is exactly $\mathrm{AV@R}_\alpha[Z]$.

Subdifferential of $\mathrm{AV@R}_\alpha[\cdot]$

We may now compute the subdifferential of $\mathrm{AV@R}_\alpha[Z]$. Assume $t^* = t^{**}$. Then a $Y \in \partial(\mathrm{AV@R}_\alpha)[Z]$ must satisfy

$$\begin{aligned} \mathbb{E}[Y] &= 1 \\ Z > t^* &\Rightarrow Y = \alpha^{-1} \\ Z < t^* &\Rightarrow Y = 0 \\ Z = t^* &\Rightarrow Y \in [0, \alpha^{-1}] \end{aligned}$$

Computation of $\mathrm{AV@R}_\alpha$

```
function [a, mu] = avar(Z, p, alpha)
% Z     : Discrete values of RV
% p     : probabilities
% a     : Average value at risk (level alpha)
% mu    : A subgradient of AVAR_alpha at Z
[mu, a, exitflag] = linprog(-Z',[],[], ...
   ones(1, n), 1, zeros(n,1), p/alpha);
assert(exitflag == 1, 'numerical problems');
a = -a;
```

Computation of $\mathrm{AV@R}_\alpha$

```
function a = avar(Z, p, alpha)
% Computation using the definition
% This code does not return a subgradient
n = length(Z);
f = [1 p/alpha];
H = -[ones(n,1) eye(n)];
[~, a]=linprog(f, H, -Z',[],[],[-Inf;zeros(n,1)]);
```

* Remark

Take $Z \geq 0$. $\mathrm{AV@R}_\alpha[Z]$ can be defined as follows[14]

$$\begin{aligned}\mathrm{AV@R}_1[Z] &= \mathbb{E}[Z \mid Z \geq \mathrm{V@R}_1[Z]] \\ &= \mathbb{E}[Z \mid Z \geq \operatorname{ess\,sup}[Z]] \\ &= \operatorname{ess\,sup}[Z],\end{aligned}$$

and $\mathrm{AVAR}_0[Z]$ is

$$\mathrm{AVAR}_0[Z] = \mathbb{E}[Z]$$

$\mathrm{AV@R}_\alpha$ can be used to bridge the distance between the **robust** ($\|\cdot\|_\infty$) and the **stochastic** ($\mathbb{E}[\cdot]$) approach.

[14] Note that $\mathrm{V@R}_1[Z] = -\infty$ and $\mathrm{V@R}_0[Z] = \operatorname{ess\,sup}[Z]$

IV. Law invariance

- ✓ Equality of distributions
- ✓ Law invaraince
- ✓ Kusuoka's Representation Theorem
- ✓ Stochastic orders

Equality of distributions

Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = 2^\Omega$ and $\mathrm{P}[\omega = \omega_i] = 1/3$. Let X be a random variable with

$$X(\omega) = \omega$$

for all $\omega \in \Omega$. Define a random variable Y with

$$Y(\omega_1) = \omega_2,$$
$$Y(\omega_2) = \omega_3,$$
$$Y(\omega_3) = \omega_1,$$

Then, X and Y have the **same distribution**, but they are **never equal**:

$$\mathrm{P}[X = \omega_i] = \mathrm{P}[Y = \omega_i] = 1/3.$$

Equality of distributions

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and $X : \Omega \to \mathbb{R}$ a $\mathcal{N}(0,1)$-distributed random variable.

Define the random variable

$$Y = -X.$$

Then, $Y \sim \mathcal{N}(0,1)$ but X and Y are a.e. unequal:

$$\mathrm{P}[X = Y] = 0.$$

Equality of distributions

Two random variables $X, Y : \Omega \to \mathbb{R}$ are **surely equal** if

$$X(\omega) = Y(\omega), \text{ for all } \omega \in \Omega.$$

They are **almost surely equal** if

$$\mathrm{P}[X \neq Y] = \mathrm{P}[\{\omega : X(\omega) \neq Y(\omega)\}] = 0$$

They are **equal in distribution**, denoted as $X \overset{d}{\sim} Y$, if

$$\mathrm{P}[X \leq c] = \mathrm{P}[Y \leq c], \text{ for all } c \in \mathbb{R}.$$

Equality of distributions

A distribution is like a musical **score**, whereas a random variable is a particular **performance**.

Equality of distributions

It is obvious that

$$X = Y \Rightarrow X \overset{\text{a.e.}}{=} Y \Rightarrow X \overset{d}{\sim} Y.$$

Law invariance

A risk measure ρ is called **law invariant** if, i.e.,

$$X \overset{d}{\sim} Y \Rightarrow \rho(X) = \rho(Y).$$

i.e., it is insensitive to how the uncertainty is produced (i.e., $X(\omega)$) and depends only on the distribution.

Law invariance importance

- Law invariance is a natural assumption for risk measures
- It is very often a necessary assumption
- Necessary assumption when studying stochastic orders
- Law invariant, coherent, lsc risk measures can be constructed using combinations of the average value-at-risk (Kusuoka theorem)

* Kusuoka Theorem

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be nonatomic and $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$ be a law-invariant, coherent, lsc risk measure. Then, there is a set $\mathfrak{M}$ of prob. measures on $(0, 1]$ so that

$$\rho(Z) = \sup_{\mu \in \mathfrak{M}} \int_0^1 \mathrm{AV@R}_\alpha[Z] \mathrm{d}\mu(\alpha),$$

for $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$.

Stochastic orders

For random variables on $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$, we may define various partial orders, known as **stochastic orders**:

- Integral stochastic orders
- Usual stochastic order
- First, second and k-th stochastic orders
- Increasing convex order
- and many many another...

Hereafter, we shall assume that $(\Omega, \mathcal{F}, \mathrm{P})$ is nonatomic.

Stochastic orders

The stochastic orders we will present here are partial orders, i.e.,

1. reflexive ($X \preceq X$)
2. transitive ($X \preceq Y$ and $Y \preceq Z$ implies $X \preceq Z$)
3. antisymmetric ($X \preceq Y$ and $Y \preceq X$ implies $Y \overset{d}{\sim} X$)

but not complete, i.e., there will be X, Y such that neither $X \preceq Y$ nor $Y \preceq X$.

Integral stochastic order

Let $\mathcal{U}$ be a collection of $\mathcal{F}$-measurable functions $u : \mathbb{R} \to \mathbb{R}$. This defines a partial order $\preceq_{\mathcal{U}}$ known as **integral stochastic order with generator $\mathcal{U}$** and is defined as

$$Z_1 \preceq_{\mathcal{U}} Z_2 \Leftrightarrow \mathbb{E}[u(Z_1)] \leq \mathbb{E}[u(Z_2)],$$

for all $u \in \mathcal{U}$.

Usual stochastic order

Taking $\mathcal{U}$ to be the class of nondecreasing functions, we have the **usual stochastic order**, that is

$$Z_1 \preceq_{(1)} Z_2 \Leftrightarrow \mathbb{E}[u(Z_1)] \leq \mathbb{E}[u(Z_2)],$$

for all for all nondecreasing functions u such that the expectations exist. This is equivalent to[15]

$$\begin{aligned} & \Phi_{Z_1}(\eta) \geq \Phi_{Z_2}(\eta), \forall \eta \in \mathbb{R} \\ \Leftrightarrow \; & \mathrm{P}[Z_1 \leq \eta] \geq \mathrm{P}[Z_2 \leq \eta] \\ \Leftrightarrow \; & \Phi_{Z_1}^{-1}(p) \leq \Phi_{Z_2}^{-1}(p), \forall p \in [0,1] \end{aligned}$$

[15] To prove this fact you can use the fact that measurable functions can be written as the pointwise limit of an increasing sequence of simple functions (exercise).

Usual stochastic order

Let $Z_1, Z_2 : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$. Then,

$$Z_1 \preceq_{(1)} Z_2$$

iff there is a probability space $(\Omega', \mathcal{F}', \mathrm{P}')$ and random variables $Z_1' \overset{d}{\sim} Z_1$, $Z_2' \overset{d}{\sim} Z_2$ so that

$$Z_1' \leq Z_2' \ \mathrm{P}'\text{-a.e.}$$

Usual stochastic order

Let $Z_1, Z_2 : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$. Then,

$$Z_1 \preceq_{(1)} Z_2$$

iff there is a probability space $(\Omega', \mathcal{F}', \mathrm{P}')$ and random variables $Z_1' \overset{d}{\sim} Z_1$, $Z_2' \overset{d}{\sim} Z_2$ so that

$$Z_1' \leq Z_2' \;\; \mathrm{P}'\text{-a.e.}$$

These are

$$Z_i' = \Phi_{Z_i}^{-1}(U),$$

where U is the uniform RV on $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$.

Usual stochastic order

If $Z_1 \preceq_{(1)} Z_2$ and $\mathbb{E}[Z_1] = \mathbb{E}[Z_2]$ then $Z_1 \overset{d}{\sim} Z_2$.

Monotonicity wrt a stochastic order

A law invariant risk measure ρ is **monotonous wrt a stochastic order** $\preceq$ if for all RVs Z_1, Z_2 we have

$$Z_1 \preceq Z_2 \Rightarrow \rho(Z_1) \leq \rho(Z_2).$$

Recall that $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \mathbb{R}$ is called **monotonous** if

$$Z_1 \leq Z_2, \text{a.e.} \Rightarrow \rho(Z_1) \leq \rho(Z_2).$$

Monotonicity wrt $\preceq_{(1)}$

Assume $(\Omega, \mathcal{F}, \mathrm{P})$ is nonatomic. Let ρ be law invariant. TFAE:

1. ρ is monotonous
2. ρ is monotonous wrt $\preceq_{(1)}$

Proof. (1⇒2) Take $Z_1 \preceq_{(1)} Z_2$, and define the RVs $Z_i \triangleq \Phi_{Z_i}^{-1}(U)$ on $(\Omega', \mathcal{F}', \mathrm{P}')$ so that $Z_i' \overset{d}{\sim} Z_i$. We have $Z_1' \leq Z_2'$ P'-*a.e.* on Ω'. Since ρ is monotonous it is $\rho(Z_1') \leq \rho(Z_2')$ and monotonicity wrt $\preceq_{(1)}$ follows because ρ is law invariant. The converse is straightforward (exercise). □

Increasing convex order

Taking $\mathcal{U}$ to be the class of nondecreasing convex functions, we have the **incresing convex order**, that is

$$Z_1 \preceq_{\text{icx}} Z_2 \Leftrightarrow \mathbb{E}[u(Z_1)] \leq \mathbb{E}[u(Z_2)],$$

for all nondecreasing convex functions u such that the expectations exist.

Increasing convex order

For two random variables $Z_1, Z_2 \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$, TFAE:

1. $Z_1 \preceq_{\text{icx}} Z_2$,
2. $\mathbb{E}[Z_1 - \eta]_+ \leq \mathbb{E}[Z_2 - \eta]_+$, for all $\eta \in \mathbb{R}$.

Monotonicity wrt $\preceq_{\text{icx}}$

Assume $(\Omega, \mathcal{F}, \mathrm{P})$ is nonatomic. Any **law invariant**, **lsc** and **coherent** risk measure $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathbb{R}}$ is monotonous wrt $\preceq_{\text{icx}}$.

Proof. (1⇒2) It is easy to show that $\mathrm{AV@R}_\alpha$ is monotonous wrt $\preceq_{\text{icx}}$. The proof follows invoking the Kusuoka representation theorem. □

Additional info cannot increase risk

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be nonatomic, $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$ and $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \bar{\mathrm{I\!R}}$ be law invariant, lsc and coherent. Then,

$$\rho(\mathbb{E}[Z \mid \mathcal{G}]) \leq \rho(Z),$$

for all $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$. Additionally,

$$\mathbb{E}[Z] \leq \rho(Z).$$

Further reading

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