Markov switching systems

Pantelis Sopasakis

IMT Institute for Advanced Studies Lucca

February 7, 2016

Outline

1. Introduction
2. Finite horizon optimal control
3. Uniform invariance
4. Lyapunov stability analysis
5. Stochastic MPC

I. Introduction

Encoding constraints in the cost

Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be the *extended real line*. We call functions of the form $F : X \to \bar{\mathbb{R}}$ (X is any set) *extended real valued* (ERV) functions. One such function is the *indicator of a set*:

$$\delta(x \mid C) = \begin{cases} 0, & \text{if } x \in C \\ +\infty, & \text{otherwise} \end{cases}$$

Encoding constraints in the cost

We can use indicator functions to encode constraints in the cost of an optimization problem. That is, let $f : \mathbb{R}^n \to \mathbb{R}$. Then,

$$\min_{x \in C} f(x),$$

is equivalent to

$$\min_{x \in \mathbb{R}^n} f(x) + \delta(x \mid C).$$

The function $F(x) = f(x) + \delta(x \mid C)$ is an extended real valued function $F : \mathbb{R}^n \to \bar{\mathbb{R}}$.

Effective domain of an ERV function

The *effective domain* of an ERV function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is the set

$$\operatorname{dom} f := \{x : f(x) < \infty\}.$$

The problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

is equivalent to

$$\min_{x \in \operatorname{dom} f} f(x).$$

Effective domain property

Assume $f, g : \mathbb{R}^n \to \bar{\mathbb{R}}$. It is then easy to verify that[1]

$$f \geq g \Rightarrow \operatorname{dom} f \subseteq \operatorname{dom} g.$$

If not, there will be $x \in \operatorname{dom} f \setminus \operatorname{dom} g$, thus $g(x) = +\infty$ while $f(x) < \infty$ which is a contradiction.

[1] The notation $f \geq g$ is meant as $f(x) \geq g(x)$ for all $x \in \mathbb{R}^n$.

Effective domain and epigraph

For an ERV function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, its epigraph a subset of $\mathbb{R}^{n+1}$ defined as follows

$$\operatorname{epi} f := \{(x, \alpha) : f(x) \leq \alpha\}.$$

Then the effective domain of f is the projection of its epigraph on the x-space, i.e.,

$$\begin{aligned} \operatorname{dom} f &= \{x \in \mathbb{R}^n : \exists \alpha \in \mathbb{R} \text{s.t. } (x, \alpha) \in \operatorname{epi} f\} \\ &= \operatorname{proj}_x \operatorname{epi} f. \end{aligned}$$

Domain of a multi-valued function

For a $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ we define its domain as the set

$$\operatorname{dom} F := \{x : F(x) \neq \varnothing\}.$$

The graph of F is defined as

$$\operatorname{gph} F := \left\{(x, y) \in \mathbb{R}^{m+n} : y \in F(x)\right\}.$$

Then, it is

$$\operatorname{dom} F = \operatorname{proj}_x \operatorname{gph} F.$$

Level boundedness

Take a function $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$

$$\ell : \mathbb{R}^n \times \mathbb{R}^m \ni (x, u) \mapsto f(x, u) \in \bar{\mathbb{R}}.$$

We say that ell is **level-bounded in** u **locally uniformly in** x if for every $\bar{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ there exists a neighbourhood of $\bar{x}$, $\mathcal{W}_{\bar{x}}$ and a bounded set $B \subset \mathbb{R}^m$ such that

$$\{u | f(x, u) \leq \alpha\} \subseteq B,$$

for all $x \in \mathcal{W}_{\bar{x}}$.

Markovian switching systems

We will work with systems of the following form:

$$x(k+1) = f_{\theta(k)}(x(k), u(k)),$$

where $\theta(k)$ is a Markovian stochastic process with values drawn from $\mathcal{N}$. When

$$f_i(x, u) = A_i x + B_i u,$$

we have a MJLS.

We assume that for all $i \in \mathcal{N}$, $f_i(\cdot, \cdot)$ are **continuous** in $\mathbb{R}^n \times \mathbb{R}^m$ and $f_i(0,0) = 0$.

Switching paths

Let $i \in \mathcal{N}$. We call the **cover** of mode i the set

$$\mathscr{C}(i) = \{j \in \mathcal{N} : p_{ij} > 0\}$$

A sequence of modes $\{i_0, i_1, \ldots\}$ is called an **admissible switching path** if $i_{s+1} \in \mathscr{C}(i_s)$ for all $s = 1, 2, \ldots$.

We denote the set of all admissible switching paths by $\mathfrak{A}$ and $\mathfrak{A}_N$ will be the set of switching paths of length N.

We also define $\mathfrak{A}(i) = \{a \in \mathfrak{A} : a_0 = i\}$ and $\mathfrak{A}_N(i) = \{a \in \mathfrak{A}_N : a_0 = i\}$.

Switching paths

Summarizing:

$$\mathfrak{A} := \{a = \{a_i\}_{i\in\mathbb{N}} \mid \mathscr{C}(a_k) \ni a_{k+1}, \forall k \in \mathbb{N}\},$$

$$\mathfrak{A}_N := \{a = \{a_i\}_{i=0}^{N} \mid \mathscr{C}(a_k) \ni a_{k+1}, \forall k = 0, \dots, N-1\},$$

and for $i \in \mathcal{N}$:

$$\mathfrak{A}(i) = \{a \in \mathfrak{A} : a_0 = i\},$$

$$\mathfrak{A}_N(i) = \{a \in \mathfrak{A}_N : a_0 = i\}.$$

Control laws and policies

A measurable function

$$\mu : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$$

is called a **control law**.

A (finite of infinite) sequence of control laws

$$\pi = \{\mu_0, \mu_1, \dots\},$$

where μ_k is $\mathcal{G}_k$-measurable[2], is called a **control policy**.

Π is the set of policies and Π_N is the set of policies of length N.

[2] Recall that $\mathcal{G}_k$ denotes the σ-algebra generated by $\{x(t), \theta(t); t = 0, \dots, N-1\}$. $\mathcal{G}_k$-measurability implies that μ_k is a function of $x(k)$ and $\theta(k)$.

Solutions of Markovian switching systems

The **solution** of the aforementioned Markovian switching system with $x(0) = x_0$, $\theta(0) = i$ following a switching path $a \in \mathfrak{A}(i)$ and using a policy $\pi \in \Pi$ is denoted by

$$\phi(k; x, i, \pi, a).$$

We have

$$\phi(k+1; x, i, \pi, a) = f_{a_k}(x_k, u_k),$$

where $x_k = \phi(k+1; x, i, \pi, a)$ and $u_k = \mu_k(x_k)$.

II. Finite horizon optimal control

Coming up...

1. Problem statement
2. Dynamic programming operators
3. Monotonicity properties of DP operators

The class of cost functions

We introduce the class of cost functions

$$\mathrm{fcns}(\mathbb{R}^n, \mathcal{N}) := \{f : \mathbb{R}^n \times \mathcal{N} \to \bar{\mathbb{R}} : f \geq 0, f(0, i) = 0, \forall i \in \mathcal{N}\}$$

FHOC problem

Let $\ell \in \mathrm{fcns}(\mathbb{R}^{n+m}, \mathcal{N})$ be the **stage cost** function – it has the form $\ell(x, u, i)$ – and $V_f \in \mathrm{fcns}(\mathbb{R}^{n}, \mathcal{N})$ be the **terminal cost** function. The finite horizon cost of a policy $\pi_N \in \Pi_N$ is

$$V_N(x, i, \pi_N) := \mathbb{E}\left[\sum_{k=0}^{N-1} \ell(x(k), u(k), \theta(k)) + V_f(x(N), \theta(N))\right]$$

with $x(0) = x$, $x(k) = \phi(k; x, i, \pi, \theta)$, $\theta \in \mathfrak{A}(i)$, $u(k) = \mu(x(k), \theta(k))$.

FHOC problem

We can of course encode constraints into the cost function V_N. In particular

$$V_N(x, i, \pi) < \infty \Leftrightarrow \begin{cases} (x(k), u(k)) \in \operatorname{dom} \ell(\cdot, \cdot, \theta(k)), \\ x(N) \in \operatorname{dom} V_f(\cdot, \theta(N)), \\ \text{for all paths } \{\theta(k)\}_{k=0,\dots,N} \in \mathfrak{A}(i). \end{cases}$$

Let $Y_i := \operatorname{dom} \ell(\cdot, \cdot, i)$ and $X_i^f := \operatorname{dom} V_f(\cdot, i)$.

Hereafter, we shall assume that the following constraints are imposed:

$$(x(k), u(k)) \in Y_{\theta(k)}, \text{ and } x_N \in X_{\theta(N)}^f.$$

FHOC problem

The **value function** is the mapping $V_N : \mathbb{R}^n \times \mathcal{N} \to \bar{\mathbb{R}}$:

$$V_N^\star(x, i) := \inf_{\pi \in \Pi_N} V_N(x, i, \pi).$$

The **optimal policy mapping** is a mapping $\Pi_N^\star : \mathbb{R}^n \times \mathcal{N} \rightrightarrows \Pi_N$

$$\Pi_N^\star := \operatorname*{arg\,min}_{\pi \in \Pi_N} V_N(x, i, \pi).$$

Dynamic programming operators

For $V \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ and control law $\mu : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$ we define

$$\begin{aligned}\mathbf{T}_\mu V(x,i) &:= \ell(x,\mu(x,i),i) + \mathbb{E}\left[V(x(k+1)) \mid \mathcal{G}_k\right] \\ &= \ell(x,\mu(x,i),i) + \mathbb{E}\left[V(x(k+1)) \mid x(k) = x, \theta(k) = i\right] \\ &= \ell(x,\mu(x,i),i) + \sum_{j \in \mathscr{C}(i)} p_{ij} V(f_i(x,\mu(x,i)),j)\end{aligned}$$

This can be seen as a function $H(x,i,\mu,V)$ for which a standard **monotonicity assumption** holds (next slide).

Monotonicity of H and T_μ

Fix $x \in \mathbb{R}^n$, $i \in \mathcal{N}$, a control law control law $\mu : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$ the following holds[3]

$$V \leq V' \Rightarrow H(x, i, \mu, V) \leq H(x, i, \mu, V'),$$

with $V, V' \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$. This readily implies that

$$V \leq V' \Rightarrow \mathbf{T}_\mu(V) \leq \mathbf{T}_\mu(V').$$

[3]For two functions $V_1, V_2 : X \to \bar{\mathbb{R}}$ (X is any set), the notation $V_1 \leq V_2$ means that for every $x \in X$ it is $V_1(x) \leq V_2(x)$.

Dynamic programming operators

Recall the definition of $\mathbf{T}_\mu$

$$\mathbf{T}_\mu V(x,i) = \ell(x,\mu(x,i),i) + \sum_{j\in\mathscr{C}(i)} p_{ij} V(f_i(x,\mu(x,i)),j).$$

The **DP operator** is defined as

$$\mathbf{T}V(x,i) := \inf_u \ell(x,u,i) + \sum_{j\in\mathscr{C}(i)} p_{ij} V(f_i(x,u),j),$$

and the **optimal control operator** is

$$\mathbf{S}V(x,i) := \arg\min_u \ell(x,u,i) + \sum_{j\in\mathscr{C}(i)} p_{ij} V(f_i(x,u),j).$$

$\mathbf{T}^k$ properties

For every $V \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$, $\mathbf{T}^k V \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ for all $k \in \mathbb{N}$.

Proof.

Recall that

$$\mathbf{T}V(x,i) = \inf_u H(x,i,u,V).$$

Since $H(0,i,0,V) = 0$ for every $V \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ and $i \in \mathcal{N}$ we have $\mathbf{T}V(0,i) = 0$. It is $H(x,i,u,V) \geq 0$ for all V and i, therefore $\mathbf{T}V \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$. □

Monotonicity of T

From the monotonicity property of H we can infer that for functions $V, V' \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ it is

$$V \leq V' \Rightarrow \mathbf{T}(V) \leq \mathbf{T}(V').$$

Let $\mathbf{T}^k$ be the composition of $\mathbf{T}$ with itself k times. Then, by induction

$$V \leq V' \Rightarrow \mathbf{T}^k(V) \leq \mathbf{T}^k(V').$$

DP solution

Let $\pi = \{\mu_i\}_{i=0}^{k}$ be a finite policy. We define $\mathbf{T}_{\mu_0}\mathbf{T}_{\mu_1}\cdots\mathbf{T}_{\mu_k}$ to be the composition of those operators. Then, using the definitions:

$$V_N(x, i, \pi) = (\mathbf{T}_{\mu_0}\cdots\mathbf{T}_{\mu_N})V_f(x, i).$$

And the value function is

$$V_N^{\star}(x, i) = \mathbf{T}^N V_f(x, i).$$

This last equation gives rise to the DP recursion.

DP algorithm

We construct a *sequence of functions* $\{V_i^\star\}_{i=0,\dots,N}$ with

$$V_0^\star = V_f$$

and

$$\begin{aligned} V_{k+1}^\star &= \mathbf{T} V_k^\star, \\ \mathcal{U}_{k+1}^\star &= \mathbf{S} V_k^\star. \end{aligned}$$

This returns $V_N^\star$ and $\Pi_N^\star \equiv \mathcal{U}_N^\star$.

DP algorithm

If we replace $\mathbf{T}$ and $\mathbf{S}$ with their definitions we retrieve the typical DP algorithm formulation

$$V_0^\star(x,i) = V_f(x,i),$$

and

$$\begin{aligned} V_{k+1}^\star(x,i) &= \inf_u \left\{ \ell(x,u,i) + \sum_{j\in\mathscr{C}(i)} p_{ij} V_k^\star(f_i(x,u),j) \right\}, \\ \mathcal{U}_{k+1}^\star(x,i) &= \arg\min_u \left\{ \ell(x,u,i) + \sum_{j\in\mathscr{C}(i)} p_{ij} V_k^\star(f_i(x,u),j) \right\}. \end{aligned}$$

DP algorithm

Assume that[4]

$$V_f \geq \mathbf{T} V_f.$$

Then,

$$V_k^\star = \mathbf{T}^{k-1} V_f \geq \mathbf{T}^k V_f = V_{k+1}^\star.$$

[4] Juxtapose with Assumption 2.12 (Basic stability assumption): J.B. Rawlings and D.Q. Mayne, Model predictive control: stability and optimality, Nob Hill Publishing, 2009.

Normality assumptions

Hereafter, we assume that for every $i \in \mathcal{N}$

1. $\ell(\cdot, \cdot, i)$ are **level-bounded** in u locally uniformly in x,
2. $V_f(\cdot, i)$ are **lower-semicontinuous**.

Consequences of the assumptions

Because of the normality assumptions:

1. $\mathbf{T}^k V_f$ is lsc for all k,
2. $\operatorname{dom} \mathcal{U}_k^\star = \operatorname{dom} V_k^\star$,
3. When the infimum is finite, it is also attained,
4. Every $\mathcal{U}_k^\star(\cdot, i)$ is compact.

III. Invariance notions for Markovian systems

Next slides...

1. Definition of a preimage operator
2. Definition of uniform control invariance (UCI)
3. Criteria for UCI
4. Link between DP and UCI
5. Maximal UCI and algorithmic determination

Collections of sets

We introduce the following definition for families of sets

$$\mathrm{sets}(\mathbb{R}^n, \mathcal{N}) := \{C = \{C_i\}_{i \in \mathcal{N}} \mid 0 \in C_i \subseteq \mathbb{R}^n, \forall i \in \mathcal{N}\}.$$

The preimage operator

For $C \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$ and $i \in \mathcal{N}$ we define

$$R(C,i) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists u \in \mathbb{R}^m : (x,u) \in Y_i, \\ f_i(x,u) \in \bigcap_{j \in \mathscr{C}(i)} C_j \end{array} \right\}$$

We can write $R(C,i)$ using the projection operator

$$R(C,i) := \mathrm{proj}_x \left\{ (x,u) \in Y_i \;\middle|\; f_i(x,u) \in \bigcap_{j \in \mathscr{C}(i)} C_j \right\}$$

Then define $R(C) \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$

$$R(C) = \{R(C,i)\}_{i \in \mathcal{N}}.$$

Understanding $R(C, i)$

Assume $Y_i = X_i \times U_i$, i.e., constraints are for the form $x(k) \in X_{\theta(k)}$ and $u(k) \in U_{\theta(k)}$.

For $C \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$ and $i \in \mathcal{N}$, $R(C, i)$ is the set of states $x \in X_i$ for which with some input $u(x) \in U_i$ so that the next state $x^+ = A_i x + B_i u$ is in all C_j with $j \in \mathscr{C}_i$.

Understanding $R(C, i)$

Let $V \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ and $\mathrm{dom}\, V = C$ (i.e., $\mathrm{dom}\, V(\cdot, i) = C_i$ and $C = \{C_i\}_{i \in \mathcal{N}}$). Then $\mathrm{dom}\, \mathbf{T}V = R(C)$.

Proof.
Fix $i \in \mathcal{N}$:

$$\begin{aligned}
\mathrm{dom}\, \mathbf{T}V(\cdot, i) &= \{x : \exists \alpha \in \mathbb{R}, (x, \alpha) \in \mathrm{epi}\, \mathbf{T}V(\cdot, i)\} \\
&= \{x : \exists \alpha \in \mathbb{R}, \exists u : (x, u, \alpha) \in \mathrm{epi}\, H(\cdot, i, \cdot, V)\} \\
&= \{x : \exists u : (x, u) \in \mathrm{dom}\, H(\cdot, i, \cdot, V)\} \\
&= R(C, i).
\end{aligned}$$

Note: We used Prop. 1.18 in: R.T. Rockafellar and R.J.B Wets, Variational Analysis, Springer, 2009. □

Properties of R

Recall that $X^f := \operatorname{dom} V_f = \operatorname{dom} V_0^\star$. What is $R(X^f)$?

$$R(X^f, i) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists u \in \mathbb{R}^m : (x,u) \in Y_i, \\ f_i(x,u) \in \bigcap_{j \in \mathscr{C}(i)} X_j^f \end{array} \right\}$$

But, using that fact that $Y_i = \operatorname{dom} \ell(\cdot, \cdot, i)$, it is

$$R(X^f, i) = \operatorname{dom} V_1^\star(\cdot, i),$$

and consequently

$$R(X^f) = \operatorname{dom} V_1^\star.$$

Properties of R

By induction, we can see that

$$R^k(X^f) = \operatorname{dom} V_k^\star.$$

and of course

$$R^N(X^f) = \operatorname{dom} V_N^\star.$$

Further Properties of R

Under the normality assumptions, it can be shown that $\operatorname{dom} V_k^\star = \operatorname{dom} \mathcal{U}_k^\star$.

$$R^k(X^f) = \underbrace{\operatorname{dom} V_k^\star = \operatorname{dom} \mathcal{U}_k^\star}_{\text{The minimum is attained.}}.$$

Uniform control invariance

A $C \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$ is called **uniformly control invariant** (UCI) for our Markovian switching system if there exists a policy $\pi \in \Pi$ such that

$$x(0) \in C_{\theta(0)} \Rightarrow \phi(k, x, \theta(0), \pi, \theta) \in C_{\theta(k)},$$

for every admissible switching path $\theta \in \mathfrak{A}(\theta(0))$.

Criterion for UCI

A $C \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$ is UCI if and only if

$$C \subseteq R(C).$$

Proof.

Hint: Assume there is a $x \in C_i$ with $x \notin R(C, j)$ for some $j \in \mathscr{C}(i)$ and for all u such that $(x, u) \in Y_i$ which leads to contradiction (Exercise). $\square$

DP and UCI

If $V_f \geq \mathbf{T}V_f$, then for $k \geq 1$: $\operatorname{dom} V_k^\star$ is UCI.

DP and UCI

If $V_f \geq \mathbf{T}V_f$, then for $k \geq 1$: $\operatorname{dom} V_k^\star$ is UCI.

Proof.

Recall that $C \in \operatorname{sets}(\mathbb{R}^n, \mathcal{N})$ is UCI iff $C \subseteq R(C)$. We know that $\operatorname{dom} V_k^\star = R^k(X^f)$ with $X^f := \operatorname{dom} V_f$. Given that $V_f \geq \mathbf{T}V_f$ we have $V_k^\star \geq V_{k+1}^\star$, thus for $k \geq 1$

$$\begin{aligned} V_k^\star \geq V_{k+1}^\star &\Rightarrow \operatorname{dom} V_k^\star \subseteq \operatorname{dom} V_{k+1}^\star \\ &\Rightarrow R^k(X^f) \subseteq R^{k+1}(X^f) \\ &\Rightarrow R^k(X^f) \subseteq R(R^k(X^f)) \\ &\Rightarrow \operatorname{dom} V_k^\star \subseteq R(\operatorname{dom} V_k^\star) \end{aligned}$$

so $\operatorname{dom} V_k^\star$ is UCI. □

Maximal UCI

Definition.
A UCI $X^\star \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$ is called a **maximal UCI** family of sets if $X^\star \supseteq X$ for every $X \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$ which is UCI.

Maximal UCI

The maximal UCI family of sets $X^\star = \{X_i^\star\}_{i \in \mathcal{N}}$ is given by

$$X_i^\star = \left\{ x \;\middle|\; \begin{array}{l} \exists \pi \in \Pi : \phi(k, x, i, \pi, \theta) \in X_{\theta(k)}, \\ \forall k \in \mathbb{N}, \forall \theta \in \mathfrak{A}(i) \end{array} \right\}.$$

Proof.

The proof is left as an exercise. □

Determination of the maximal UCI

Assume that constraints are given in the form $x(k) \in X_{\theta(k)}$ and $u(k) \in U_{\theta(k)}$. The following recursion converges to the maximal UCI:

$$X^0 = X,$$
$$X^{k+1} = R(X_k),$$

where $X = \{X_i\}_{i \in \mathcal{N}}$ and notice that

$$X_i^k = \left\{ x \;\middle|\; \begin{array}{l} \exists \pi \in \Pi : \phi(k, x, i, \pi, \theta) \in X_{\theta(k)}, \\ \forall k \in \mathbb{N}, \forall \theta \in \mathfrak{A}_k(i) \end{array} \right\}.$$

But, the algorithm must converge in **finitely many steps** to return the maximal UCI...

Determination of the maximal UCI

Using the procedure:

$$X^0 = X,$$
$$X^{k+1} = R(X_k),$$

Assume X is a polytope and the system is a MJLS.

+ We converge to the **maximal** UCI
- Termination: $X^k = X^{k-1}$. It may not converge in **finite time** (even for MSS systems).
- It is computationally expensive to compute the minimal representation of each X^k
- None of the iterates needs to be UCI.

IV. Lyapunov stability analysis

Coming up...

1. Uniform positive invariance
2. Definitions: MSS and MSES
3. Uniform positive invariance
4. Lyapunov stability conditions

Autonomous Markovian systems

Consider the Markovian switched system

$$x(k+1) = f^{\mu}_{\theta(k)}(x(k)) := f_{\theta(k)}(x(k), \mu(x(k), \theta(k))).$$

The solution of this system with $x(0) = x$, $\theta(0) = i$ and $\theta \in \mathfrak{A}(i)$ is given by

$$x(k) = \phi(k, x, i, \theta).$$

We shall assume that the system state must satisfy $x(k) \in X_{\theta(k)}$.

Uniform positive invariance

A $C \in \operatorname{sets}(\mathbb{R}^n, \mathcal{N})$ is **uniformly positively invariant** if there exists $\pi \in \Pi$ so that $\phi(k, x, i, \theta) \in C_{\theta(k)}$ whenever $x(0) = x \in C_{\theta(0)}$ for all $\theta \in \mathfrak{A}(i)$.

Now the preimage operator becomes

$$R(C, i) = \left\{ x \in X_i : f_i^{\mu}(x) \in \bigcap_{j \in \mathscr{C}(i)} C_j \right\}$$

Let $R(C) = \{R(C, i)\}_{i \in \mathcal{N}}$. Then X is UPI iff $C \subseteq R(C)$.

UPI determination

The maximal UPI set can be computed using the preimage iteration (same as for UCI) which converges in **finite steps** if and only if the closed-loop system is **uniformly asymptotically stable**.

MSS for constrained systems

Stability makes sense only with respect to a **UPI** set!

MSS for constrained systems

Let $X \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$ be a uniformly positive invariant family of sets for $x(k+1) = f^{\mu}_{\theta(k)}(x(k))$. The origin is called **mean square stable** if

$$\mathbb{E}\left[\|\phi(k, x, i, \theta)\|^2\right] \to 0, \text{ as } k \to \infty,$$

for all $x \in X_i$ and $i \in \mathcal{N}$.

MSES for constrained systems

Let $X \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$ be a UPI for $x(k+1) = f^{\mu}_{\theta(k)}(x(k))$. The origin is called **mean square exponentially stable** if there exist $\beta > 1$ and $\eta \in (0,1)$

$$\mathbb{E}\left[\|\phi(k,x,i,\theta)\|^2\right] \leq \beta \zeta^k \|x\|^2,$$

for all $x \in X_i$ and $i \in \mathcal{N}$.

Definition of $\mathcal{L}V$

For $V \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ define the operator

$$\mathcal{L}V(x(k), \theta(k)) := \mathbb{E}\left[V(x(k+1), \theta(k+1)) - V(x(k), \theta(k)) \mid \mathcal{G}_k\right].$$

It is easier to remember it as

$$\mathcal{L}V(x, i) := \mathbb{E}\left[V(x^+, i^+) - V(x, i) \mid (x, i): \text{ given}\right].$$

This can be written as

$$\begin{aligned}\mathcal{L}V(x(k), \theta(k)) &:= \mathbb{E}\left[V(x(k+1), \theta(k+1)) \mid \mathcal{G}_k\right] - V(x(k), \theta(k)) \\ &= \sum_{j \in \mathscr{C}(i)} p_{ij} V(f^{\mu}_{\theta(k)}(x(k)), \theta(k)) - V(x(k), \theta(k)).\end{aligned}$$

Lyapunov theorem for MSS

If there is a $V \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ and a $\gamma > 0$ so that

$$\mathcal{L}V(x, i) \leq -\gamma \|x\|^2,$$

for all $x \in X_i$ and $i \in \mathcal{N}$, then the origin is **MSS**[5].

[5] For details and proofs see: Patrinos et al., 2014.

Lyapunov theorem for MSS

If there is a $V \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ and a $\alpha, \beta, \gamma > 0$ so that

$$\begin{aligned} \mathcal{L}V(x,i) &\leq -\gamma \|x\|^2, \\ \alpha \|x\|^2 \leq V(x,i) &\leq \beta \|x\|^2, \end{aligned}$$

for all $x \in X_i$ and $i \in \mathcal{N}$, then the origin is **MSES**.

* Supermartingale property

A $\mathcal{L}^1(\Omega, \mathcal{F}, \mathrm{P})$-random process $\{\xi_k\}_k$ which is adapted to a filtration $\{\mathcal{F}_k\}_k$ is called a **supermartingale** if

$$\mathbb{E}[\xi_{k+1} \mid \mathcal{F}_k] \leq \xi_k, \text{ w.p. } 1$$

for all $k \in \mathbb{N}$, $k \geq 1$.

* Supermartingale property

The condition $\mathcal{L}V(x,i) \leq -\gamma\|x\|^2$ implies that $\{V(x_k, i_k)\}_k$ is a supermatringale:

$$
\begin{aligned}
&\mathcal{L}V(x_k, i_k) \leq -\gamma\|x_k\|^2 \\
\Leftrightarrow\ &\mathbb{E}[V(x_{k+1}, i_{k+1}) - V(x_k, i_k) \mid \mathcal{G}_k] \leq -\gamma\|x_k\|^2 \\
\Leftrightarrow\ &\mathbb{E}[V(x_{k+1}, i_{k+1}) \mid \mathcal{G}_k] - V(x_k, i_k) \leq -\gamma\|x_k\|^2 \\
\Rightarrow\ &\mathbb{E}[V(x_{k+1}, i_{k+1}) \mid \mathcal{G}_k] \leq V(x_k, i_k)
\end{aligned}
$$

$\therefore$ We can invoke Doob's convergence theorem!

* Doob's convergence theorem

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k\in\mathbb{N}}, \mathrm{P})$ be a filtered probability space and $\{Z_k\}_k$ an $\mathcal{F}_k$-adapted supermartingale satisfying

$$\sup_{k\in\mathbb{N}} \mathbb{E}[|Z_k|] < \infty.$$

Then the limit $Z_\infty = \lim_{k\to\infty} Z_k$ exists almost surely and $\mathbb{E}[Z_\infty] < \infty$.

V. Stochastic MPC

1. The receding horizon control law
2. Stability conditions
3. Stabilising MPC for constrained MJLS

Receding horizon control

The receding horizon control policy consists in solving the FHOC problem and applying the *first* control action to the system, that is[6]

$$u(k) = \mu_N^\star(x(k), \theta(k)),$$

where

$$\mu_N^\star(x, i) \in \mathcal{U}_N^\star(x, i).$$

The controlled system will then be

$$x(k+1) = f_{\theta(k)}^{\mu_N^\star}(x(k)).$$

[6]We will refer to this control law as the *stochastic MPC* control law.

Stochastic MPC stability

Assume that $V_f \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ is lsc and $\mathbf{T}V_f \leq V_f$ and there is an $\alpha > 0$ s.t. $\ell(x,u,i) \geq \alpha\|x\|^2$ for all $i \in \mathcal{N}$, $(x,u) \in Y_i$. Then the origin of the MPC-controlled system is MSS in $X^\star := \mathrm{dom}\, V_N^\star$.

Proof.

We will show that $V_N^\star$ is a Lyapunov function. We have $V_N^\star \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$. It is $V_N^\star = \mathbf{T}_{\mu_N^\star} V_{N-1}$, that is

$$V_N^\star(x,i) = \ell(x, \mu_N^\star(x), i) + \sum_{j \in \mathscr{C}(i)} p_{ij} V_{N-1}^\star(f_i^{\mu_N^\star}(x), j).$$

By definition, we have

$$\mathcal{L}V_N^\star(x,i) = \sum_{j \in \mathcal{N}} p_{ij} V_N^\star(f_i^{\mu_N^\star}(x), j) - V_N^\star(x,i)$$

Stochastic MPC stability

Proof (cont'd).

Let us now plug $V_N^\star$ into $\mathcal{L}V_N^\star$.

$$\mathcal{L}V_N^\star(x,i) = \sum_{j\in\mathcal{N}} p_{ij} V_N^\star(f_i^{\mu_N^\star}(x), j) - \ell(x, \mu_N^\star(x), i) - \sum_{j\in\mathscr{C}(i)} p_{ij} V_{N-1}^\star(f_i^{\mu_N^\star}(x), j)$$

But given that $\mathbf{T}V_f \leq V_f$, we have $V_N^\star \leq V_{N-1}^\star$, so

$$\mathcal{L}V_N^\star(x,i) \leq -\ell(x, \mu_N^\star(x), i) \leq -\alpha\|x\|^2,$$

which proves MSS. □

Examples of choosing V_f #0

A trivial choice is

$$V_f(x, i) = \delta_{\{0\}}(x),$$

but then $\operatorname{dom} V_N^\star$ shouldn't be expected to be too large.

Examples of choosing V_f #1

For MJLS, we can choose V_f to be

$$V_f(x,i) = x' P_i x + \delta_{X_i^f}(x),$$

where $P = (P_i)_i$ solves the CARE

$$P_i = A_i' \mathcal{E}_i(P) A_i - A_i \mathcal{E}_i(P) B_i (R_i + B_i' \mathcal{E}_i(P) B_i)^{-1} B_i' \mathcal{E}_i(P) A_i + Q_i,$$

and $X^f = \{X_i^f\}_{i \in \mathcal{N}}$ is the maximal uniformly pos. invariant set for the closed-loop system with $\mu(x,i) = F_i(P)x$.

Examples of choosing V_f #2

For MJLS: Assume that ℓ is piecewise quadratic, i.e.,

$$\ell(x,u,i) = x'Q_i x + u'R_i u + \delta_{Y_i}(x,u).$$

If we require that V_f has the following form

$$V_f(x,i) = x'P_i x + \delta_{X_i^f}(x),$$

then for $V_f \geq \mathbf{T}V_f$ to hold it is necessary that

$$\begin{aligned} \operatorname{dom} V_f \subseteq \operatorname{dom} \mathbf{T}V_f &\Leftrightarrow X^f \subseteq R(X^f) \\ &\Leftrightarrow X^f \text{ is UCI} \end{aligned}$$

and...

Examples of choosing V_f #2

for all $x \in \operatorname{dom} V_f(\cdot, i)$

$$V_f(x,i) \geq \mathbf{T}V_f(x,i) := \inf_u \ell(x,u,i) + \sum_{j \in \mathscr{C}(i)} p_{ij} V(A_i x + B_i u, j).$$

This inequality will be satisfied if there is a control law $u(x,i) = K_i x$ s.t.

$$\begin{aligned} V_f(x,i) &\geq \ell(x, K_i x, i) + \sum_j p_{ij} V_f((A_i + B_i K_i)x, j) \\ &= x' \left[Q_i + K_i' R_i K_i + (A_i + B_i K_i)' \mathcal{E}(P)(A_i + B_i K_i) \right] x \\ &\quad + \delta_{Y_i}(x, K_i x) + \sum_{j \in \mathscr{C}(i)} \delta_{X_j^f}((A_i + B_i K_i)x). \end{aligned}$$

Examples of choosing V_f #2

We can pick a UCI set $X^f \in \mathrm{sets}(\mathbb{R}^n, \mathcal{N})$, a control law $u(x,i) = K_i x$, and a PWQ stage cost $\ell(x,u,i) = x'Q_i x + u'R_i u + \delta_{Y_i}(x,u)$ with $Q_i = Q_i' \geq 0$, $R_i = R_i' > 0$, so that

$$\begin{aligned} (x, K_i x) &\in Y_i, \forall x \in X_i^f, \forall i \in \mathcal{N}, \\ (A_i + B_i K_i)x &\in X_j^f, \forall j \in \mathscr{C}(i), \forall x \in X_i^f, \forall i \in \mathcal{N}, \\ P_i &\geq Q_i + K_i' R_i K_i + (A_i + B_i K_i)' \mathcal{E}(P)(A_i + B_i K_i), \forall i \in \mathcal{N}, \\ P_i &= P_i' > 0, \forall i \in \mathcal{N}. \end{aligned}$$

Then, the MPC-controlled system is MSS over $X^\star = \operatorname{dom} V_N^\star$. The above can be cast as an LMI (*Exercise*).

Examples of choosing V_f #2

Notice that the first two requirements

$$(x, K_i x) \in Y_i, \forall x \in X_i^f, \forall i \in \mathcal{N},$$
$$(A_i + B_i K_i)x \in X_j^f, \forall j \in \mathscr{C}(i), \forall x \in X_i^f, \forall i \in \mathcal{N},$$

imply that X^f is **UPI** for the closed-loop system

$$x(k+1) = (A_{\theta(k)} + B_{\theta(k)} K_{\theta(k)})x(k),$$

subject to the constraints

$$(x(k), K_{\theta(k)} x(k)) \in Y_{\theta(k)}.$$

Ellipsoidal UCI sets

Assuming again that ℓ is PWQ and V_f is quadratic over X^f, we need to compute a UCI set[7]. Choose

$$X_i^f = \{x \mid x' P_i x \leq 1\},$$

where P_i satisfy the inequalities on the previous slide. Under proper conditions, this will be a UPI set for the closed-loop system

$$x(k+1) = (A_{\theta(k)} + B_{\theta(k)} K_\theta(k)) x(k).$$

[7] We can of course compute the maximal UCI set using the preimage iteration, but this may not converge and is often too cumbersome computationally especially in high-dimensional spaces. We can also use $X_i^f = \{0\}$, but then $\operatorname{dom} V_N^\star$ becomes too small.

Ellipsoidal UCI sets

A sufficient condition for X^f to be UCI is

$$x'P_i x \leq 1 \Rightarrow x'(A_i + B_i K_i)' P_j (A_i + B_i K_i) x \leq 1$$

for all $i \in \mathcal{N}$ and $j \in \mathscr{C}(i)$. This can be cast as an LMI using the S-lemma (*Exercise*). Ellipsoidal UCI sets are often easier to determine than polytopic ones.

MSES for stochastic MPC

Assume that $V_f \in \mathrm{fcns}(\mathbb{R}^n, \mathcal{N})$ is lsc and $\mathbf{T}V_f \leq V_f$ and there is an $\alpha > 0$ s.t. $\ell(x, u, i) \geq \alpha \|x\|^2$ for all $i \in \mathcal{N}$, $(x, u) \in Y_i$.

Additionally, assume that

1. $0 \in \operatorname{int} \operatorname{dom} V_f$,
2. each $V_N^\star(\cdot, i)$ is continuous on $X_i^N := \operatorname{dom} V_N(\cdot, i)$ and
3. each X_i^N is compact.

Then, the origin is **MSES** in $X^\star$ for the MPC-controlled system.

MSES-stabilising stochastic MPC

When applying a stochastic MPC to

1. a MJLS
2. with a PWQ stage cost ($Q_i = Q_i' \geq 0$ and $R_i = R_i' > 0$),
3. $V_f(x, i) = x' P_i x + \delta_{X_i^f}(x)$; P is the solution of the CARE and
4. X^f is the maximal UPI of the closed-loop system with the control law associated to the CARE,

then, the origin is **MSES** for the SMPC-controlled system.

Samuelson's macro-economic model

Samuelson's multiplier-accelerator macroeconomic model is a MJLS[8] with modes:

- Normal
- Boom
- Slump

based on the economy's **marginal propensity to save**.

The model's state is related to the **national income** and the input corresponds to the **government expenditure**.

[8] W.P. Blair and D.D. Sworder. Feedback control of a class of linear discrete systems with jump parameters and quadratic cost criteria. Int. J. Cont., 21(5):833–841, 1975.

Samuelson's macro-economic model

Three modes with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2.5 & 3.2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -4.3 & 4.5 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 5.3 & -5.2 \end{bmatrix}$$

and

$$B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with transition matrix

$$P = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{bmatrix}$$

Mode-dependent constraints:

$$Y_1 = [-10, 10]^2 \times [-10, 10],$$
$$Y_2 = [-8, 8]^2 \times [-10, 10],$$
$$Y_3 = [-12, 12]^2 \times [-10, 10].$$

Mode-dependent quadratic stage cost:

$$Q_1 = \begin{bmatrix} 3.6 & -3.8 \\ -3.8 & 4.87 \end{bmatrix}, Q_2 = \begin{bmatrix} 10 & -3 \\ -3 & 8 \end{bmatrix}, Q_3 = \begin{bmatrix} 5 & -4.5 \\ -4.5 & 5 \end{bmatrix},$$

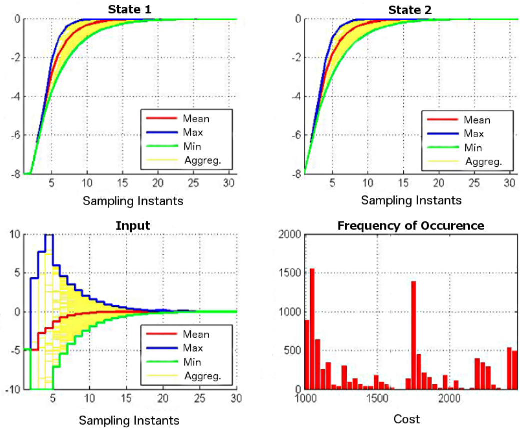
and

$$R_1 = 2.6, R_2 = 1.165, \text{ and } R_3 = 1.111.$$

Prediction horizon $N = 6$.

Samuelson's macro-economic model

10^4 random simulations with $i_0 = 2$.



VI. Conclusions

Open research questions

1. Satisfaction of constraints in probability
2. Economic stochastic MPC
3. Efficient numerical algorithms for the solution of stochastic MPC problems
4. Efficient methodologies for the computation of uniformly invariant families of sets

References

1. O.L.V. Costa, M.D. Fragoso and R.P. Marques, *Discrete-time Markov Jump Linear Systems*, Springer 2005.
2. P. Patrinos, P. Sopasakis, H. Sarimveis and A. Bemporad, Stochastic model predictive control for constrained discrete-time Markovian switching systems, Automatica 50, pp. 2504-2514, 2014.
3. A. Shapiro, D. Dentcheva, and A.P. Ruszczyński, *Lectures on stochastic programming: modeling and theory*, SIAM 2009.
4. H.J. Kushner, *Introduction to stochastic control*, Holt, Rinehart and Winston Editions, 1971.
5. R.T. Rockafellar and R.J.B Wets, *Variational Analysis*, Springer, 2009.