Economic model predictive control

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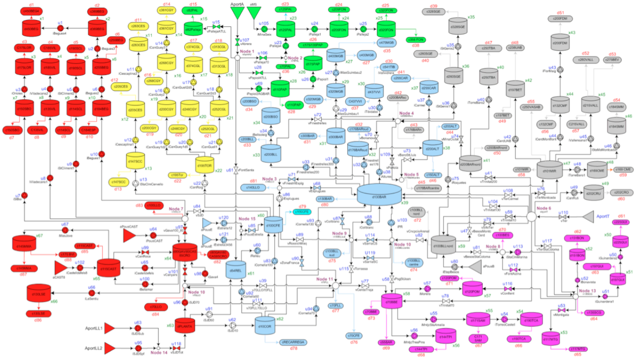
Outline

- ✓ Economic MPC – Introduction and basic properties
- ✓ Dissipativity
- ✓ Stability
- ✓ Terminal cost and constraints
- ✓ Asymptotic average constraints

I. Introduction to EMPC

Example: Control of water networks

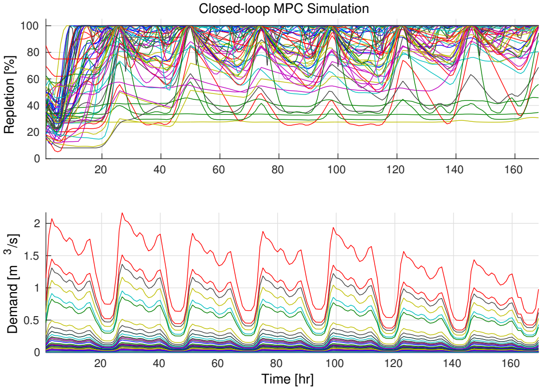


Example: Control of water networks

Control objectives:

1. Economic operation
2. Smoothness of control actions
3. Maintenance of a safety volume in each reservoir
4. Satisfaction of constraints
5. ~~Stability~~

Example: Control of water networks



Problem statement

For a dynamical system

$$x(k+1) = f(x(k), u(k)),$$

subject to the constraints $(x(k), u(k)) \in Z$ (Z: compact) for $k \in \mathbb{N}$, we compute a RH controller solving the optimisation problem

$$V^\star(x(0)) = \min_{\pi=\{u_k\}} V_N(x(0), \pi),$$

subject to the system dynamics and constraints.

What will the closed-loop system properties be?

Problem statement

The cost function $V_N : \mathbb{R}^n \times \mathbb{R}^{mN} \to \bar{\mathbb{R}}$ is assumed to have the following structure

$$V_N(x, \pi) = \sum_{k=0}^{N-1} \ell(x(k), u(k)),$$

where $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ is a lower semicontinuous, level-bounded and proper cost related to the process economics.

Stage cost

In standard MPC, the stage cost has the following property:

$$0 = \ell(x_s, u_s) = \inf_{(x,u)\in Z} \ell(x, u),$$

where (x_s, u_s) is an equilibrium point, i.e., $f(x_s, u_s) = x_s$. Typically, in MPC the stage cost looks something like

$$\ell(x, u) = \|x - x_s\|_Q^2 + \|u - u_s\|_R^2,$$

where $Q = Q' \geq 0$ and $R = R' > 0$. In economic MPC this is not assumed, i.e., there may be $(x, u) \in Z$ with[1]

$$\ell(x, u) < \ell(x_s, u_s).$$

[1]Such a (x, u) does not need to be an equilibrium point.

Stage cost

In economic MPC, the stage cost ℓ reflects the **process economics**, not a control objective.

Optimal steady states

An equilibrium pair (x_s, u_s) – $f(x_s, u_s) = x_s$ – is called optimal if[2]

$$(x_s, u_s) \in \arg\min_{x,u}\{\ell(x, u) \mid x = f(x, u), (x, u) \in Z\}.$$

Still, there may be a non-steady control-input pair $(x, u) \in Z$ so that

$$\ell(x, u) < \ell(x_s, u_s).$$

[2] Under the prescribed assumptions the minimum is attained.

EMPC formulation

The simplest version of EMPC is the following

$$\begin{aligned} V_N^\star(x) = &\min_{\pi} V_N(x,\pi) \\ \text{s.t. } & x(k+1) = f(x(k),u(k)), \forall k \in \mathbb{N}_{[0,N-1]} \\ & (x(k),u(k)) \in Z, \forall k \in \mathbb{N}_{[0,N-1]} \\ & x(N) = x_s, \\ & x(0) = x. \end{aligned}$$

Denote by $U_N^\star(x) \subseteq \mathbb{R}^{mN}$ the corresponding minimizer (need not be unique). It is $U_N^\star(x) = \{U_0^\star(x), U_1^\star(x), \ldots, U_{N-1}^\star(x)\}$

How will the solution of $U_N^\star(x_s)$ look like?

Strolling around...

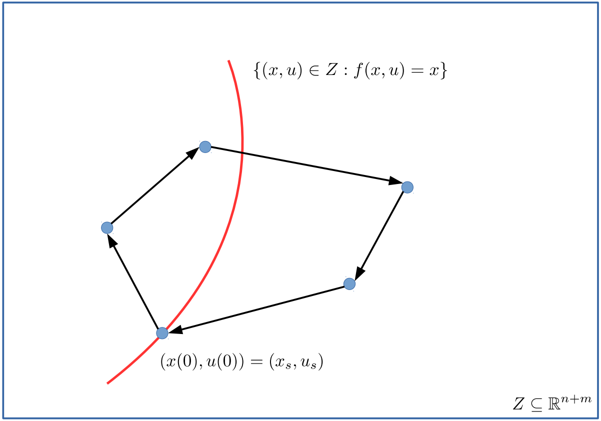


Figure : The optimal trajectory starting from x_s at time 0 and returning to x_s at time N may move around.

Some definitions

Define

$$Z_N := \left\{ (x,\pi) \;\middle|\; \begin{array}{l} \exists x(1),\ldots,x(N), x(k+1) = f(x(k),u(k)), \\ (x(k),u(k)) \in Z, \forall k \in \mathbb{N}_{[0,N-1]}, \\ x(N) = x_s, x(0) = x \end{array} \right\}$$
$$= \operatorname{dom} V_N^\star,$$

and

$$X_N := \{x : \exists \pi : (x,\pi) \in Z_N\} = \operatorname{proj}_x Z_N.$$

The economic MPC optimisation problem defines a control law $u = \kappa_N(x) \in U_0^\star(x)$, $\kappa_N : X_N \to \mathbb{R}^m$ defined as the *receding horizon control law*.

Assumptions

We shall always assume that

1. Functions $f : Z \to \mathbb{R}^n$ and $\ell : Z \to \mathbb{R}$ are continuous,
2. $x_s \in \operatorname{int} X_N$
3. There exists a $\mathscr{K}_\infty$-class function $\gamma : \mathbb{R}^n \to \mathbb{R}_+$ such that for $x \in X_N$ there is a π with $(x, \pi) \in Z$ so that

$$\|\pi - (u_s, u_s, \ldots, u_s)'\| \leq \gamma(\|x - x_s\|).$$

4. Z is a compact nonempty set

Control invariance of X_N

X_N is control invariant.

Proof.
Take $x(0) = x \in X_N$; there is π such that $(x, \pi) \in Z_N$

$$\pi = (u_0, u_1, \dots, u_{N-1}),$$

then choose $\tilde{\pi}$ be $\tilde{\pi} = (u_1, \dots, u_{N-1}, u_s)$, so that for $x(1) = f(x, u_0)$, $\tilde{\pi}$ is a feasible sequence, thus $x(1) \in X_N$. $\square$

Properties of $V_N^\star$

Function $V_N^\star$ satisfies the following inequality

$$V_N^\star(f(x, \kappa_N(x)) - V_N^\star(x) \leq \ell(x_s, u_s) - \ell(x, \kappa_N(x)),$$

for all $x \in X_N$. The right-hand side is not, however, necessarily non-positive. Proof: exercise.

Caveats...

Since it is not assumed that $\tau(x) := \ell(x_s, u_s) - \ell(x, \kappa_N(x)) \prec 0$,

1. $\{V_N^\star(x_k)\}_k$ may not be **monotonically decreasing**,
2. The ∞-horizon cost may **diverge**,
3. $V_N^\star(\cdot)$ is not a **Lyapunov function** for the c/l system.

Closed-loop asymp. average performance

To assess the performance of the closed-loop system we introduce the following index:

$$J(\kappa_N) := \limsup_{T \to \infty} \frac{\sum_{k=0}^{T} \ell(x(k), \kappa_N(x(k)))}{T+1},$$

which is called *asymptotic average cost*.

Performance bounds

The EMPC-controlled system has an asymptotic average performance that is no worse than that of the best admissible steady state.

Performance bounds

The following performance bound holds:

$$J(\kappa_N) \leq \ell(x_s, u_s).$$

Proof.

Hint: use $V_N^\star(x^+) - V_N^\star(x) \leq \ell(x_s, u_s) - \ell(x, u)$, take as. averages on both sides. Assume – without loss of generality – that $\ell(x, u) \geq 0$ over Z_N. □

Note. this bound holds only for the as. average cost, i.e., for any given T we cannot prove that

$$\frac{\sum_{k=0}^{T} \ell(x(k), \kappa_N(x(k)))}{T+1},$$

is bounded by $\ell(x_s, u_s)$.

End of first section

Conclusions.

1. Stability is not always in question
2. The stage cost ℓ may reflect an economic or performance objective
3. We studied a simple EMPC formulation with $x(N) = x_s$
4. for which Recursive feasibility is guaranteed
5. Perfomance is quantified using the asymptotic average cost
6. which is bounded above by the steady-state operation as. aver. cost.

II. Dissipativity

Dissipativity

A control system is called **dissipative**[3] with respect to a *supply rate* $s : Z \to \mathbb{R}$, $Z \subseteq \mathbb{R}^{n+m}$, if there exists a function $\lambda : \mathbb{R}^n \to \mathbb{R}$ such that for all $(x, u) \in Z$

$$\lambda(f(x,u)) - \lambda(x) \leq s(x,u).$$

[3] *Dissipativity* is for open-loop systems what *stability* is for closed-loop ones.

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If, additionally, there is a pos. def. function $\rho : \mathbb{R}^n \to \mathbb{R}$ so that for all $(x, u) \in Z$

$$\lambda(f(x,u)) - \lambda(x) \leq -\rho(x) + s(x,u),$$

then the control system is called **strictly dissipative**.

[3] *Dissipativity* is for open-loop systems what *stability* is for closed-loop ones.

Dissipativity

Often we use the following supply rate

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Then, the system is dissipative if

$$\begin{aligned} &\lambda(f(x)) - \lambda(x) \leq \ell(x,u) - \ell(x_s,u_s) \\ \Leftrightarrow &\ell(x,u) + \lambda(x) - \lambda(f(x,u)) \geq \ell(x_s,u_s) \\ \Leftrightarrow &\min_{(x,u)\in Z} \left[\ell(x,u) + \lambda(x) - \lambda(f(x,u))\right] \geq \ell(x_s,u_s) \end{aligned}$$

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It will be useful to define the **rotated stage cost**

$$L(x,u) := \ell(x,u) + \lambda(x) - \lambda(f(x,u)).$$

* A little detail

On the pervious slide the following optimisation problem was formulated:

$$\min_{(x,u)\in Z} \left[\ell(x,u) + \lambda(x) - \lambda(f(x))\right].$$

But, is the $\min$ attained for *any* functions ℓ and λ?

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$$\min_{(x,u)\in Z} \left[\ell(x,u) + \lambda(x) - \lambda(f(x))\right].$$

But, is the $\min$ attained for *any* functions ℓ and λ?

The answer is affirmative when $L(x,u)$ is **lsc**, **level-bounded** (all level-sets are bounded) and **proper**.

Dissipativity

Notice that

$$\min_{(x,u)\in Z} L(x,u) \leq \min_{\substack{(x,u)\in Z \\ x=f(x,u)}} L(x,u)$$

$$= \min_{\substack{(x,u)\in Z \\ x=f(x,u)}} \ell(x,u) = \ell(x_s,u_s)$$

Dissipativity holds when:

$$\min_{(x,u)\in Z} L(x,u) \geq \min_{\substack{(x,u)\in Z \\ x=f(x,u)}} \ell(x_s,u_s).$$

Strong duality

Using again $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$ and $\lambda(x) = y'x$ the dissipativity condition becomes

$$\min_{(x,u)\in Z} \left[\ell(x,u) + y'(x - f(x,u))\right] \geq \ell(x_s,u_s)$$

$$\Leftarrow \max_{y\in\mathbb{R}^n} \min_{(x,u)\in Z} \left[\ell(x,u) + y'(x - f(x,u))\right] = \min_{\substack{(x,u)\in Z \\ x=f(x,u)}} \ell(x,u),$$

which is a **strong duality** condition, or equivalently:

$$y'(f(x,u) - x) \leq \ell(x,u) - \ell(x_s,u_s), \forall x,u$$

Dissipative but not strongly dual

A system can be dissipative wrt $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$*, but strong duality may not hold.*

Example. Linear dynamics

$$x(k+1) = \alpha x(k) + (1-\alpha)u(k), \quad \alpha \in [0,1)$$

with stage cost

$$\ell(x,u) = (x + \frac{u}{3})(2u - x) + (x-u)^4.$$

Strong duality does not hold, but using $\lambda(x) = kx^2$, we can show that for $\alpha \in [0.5, 1)$, $\exists k = k(\alpha)$ s.t. the system is dissipative wrt s.

End of second section

Conclusions.

1. Dissipativity is the open-loop counterpart of stability
2. When $\min_{(x,u} \in S\ell(x,u)$ is str. dual, then we have dissipativity wrt $s(x,u) = \ell(x,u) - \ell(x_s, u_s)$
3. Absense of strong duality does not mean that the system is not dissipative wrt s
4. We defined the rotated cost $L(x,u) = \ell(x,u) + \lambda(x) - \lambda(f(x,u))$ which will come in handy later
5. We'll use a strong dissipativity assumption to prove stability

III. Asymptotic stability

Asymptotic stability

Under what conditions is x_s an asymptotically stable equilibrium point for the EMPC-controlled system?

Asymptotic stability

Under what conditions is x_s an asymptotically stable equilibrium point for the EMPC-controlled system?

To answer this question we introduce the following auxiliary cost function:

$$\tilde{V}_N(x, \pi) = \sum_{k=0}^{N-1} L(x(k), u(k)).$$

and we formulate the auxiliary EMPC problem

$$\tilde{V}_N^\star(x) = \min_{\pi} \tilde{V}_N(x, \pi),$$

subject to the same constraints.

Equivalence of the two costs

The feasible domain of $V_N^\star(x)$ is the same as the feasible domain for $\tilde{V}_N^\star(x)$. Also, notice that

$$\begin{aligned}
\tilde{V}_N(x,\pi) &= \sum_{k=0}^{N-1} L(x(k),u(k)) \\
&= \sum_{k=0}^{N-1} \ell(x(k),u(k)) + \lambda(x(k)) - \lambda(f(x(k),u(k))) \\
&= \underbrace{\lambda(x(0)) - \lambda(x_s)}_{\text{constant}} + V_N(x,\pi),
\end{aligned}$$

so a $\pi = \pi(x)$ is a minimizer of $V_N(x,\pi)$ if and only if it is a minimizer of $\tilde{V}_N(x,\pi)$.

Asymptotic stability condition

Let (x_s, u_s) be an optimal equilibrium point. If the control system is *strictly dissipative* wrt $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$ then x_s is an asymptotically stable equilibrium point for the EMPC-controlled system with region of attraction X_N.

Proof.

Use the equivalence between $V_N^\star$ and $\tilde{V}_N^\star$ and the definition of strict dissipativity to show that $\tilde{V}_N^\star(x^+) - \tilde{V}_N^\star(x) \leq -\rho(x)$ for $x \in X_N$. □

Enforcement of stability

We can enforce asymptotic stability by modifying the stage cost

$$\bar{\ell}(x,u) = \ell(x,u) + \alpha(x,u).$$

We choose $\alpha : Z \to \mathbb{R}$ to be positive definite with respect to (x_s, u_s).

Some observations
[Let $S := \{(x,u) \in Z, f(x,u) = x\}$]

1. $\bar{\ell}(x_s,u_s) = \ell(x_s,u_s) + \alpha(x_s,u_s) = \ell(x_s,u_s)$,
2. $\ell(x,u) \geq \ell(x_s,u_s)$ for $(x,u) \in S$, (x_s,u_s) is optimal over S
3. $\alpha(x,u) \geq \alpha(x_s,u_s) = 0$, because α is PD wrt (x_s,u_s)
4. By combining the above two $\bar{\ell}(x,u) \geq \ell(x_s,u_s)$.

Enforcement of stability

To achieve strict dissipativity wrt $\bar{s}(x,u) = \bar{\ell}(x,u) - \bar{\ell}(x_s,u_s)$ the following needs to hold (we choose $\lambda(x) = y'x$)

$$\begin{aligned} &\lambda(f(x)) - \lambda(x) \le -\rho(x) + \bar{s}(x,u) \\ \Leftrightarrow\ & y'(f(x,u) - x) \le -\rho(x) + \bar{\ell}(x,u) - \bar{\ell}(x_s,u_s) \\ \Leftrightarrow\ & y'(f(x,u) - x) \le -\rho(x) + \ell(x,u) + \alpha(x,u) - \ell(x_s,u_s) \\ \Leftrightarrow\ & \alpha(x,u) \ge \rho(x) + \ell(x_s,u_s) + y'(f(x,u) - x) - \ell(x,u) \\ \Leftrightarrow\ & \alpha(x,u) \ge h(x,u,y). \end{aligned}$$

Now for $r \ge 0$ define

$$H(r,y) := \max_{x,u} \left\{ h(x,u,y) \mid (x,u) \in Z, \left\| \begin{bmatrix} x \\ u \end{bmatrix} - \begin{bmatrix} x_s \\ u_s \end{bmatrix} \right\| \le r \right\}.$$

Enforcement of stability

We can then choose α as follows

$$\alpha(x,u) := H(\| \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_s \\ u_s \end{smallmatrix}\right] \|, y_0),$$

for some $y_0 \in \mathbb{R}^n$.

Terminal region and cost

A more flexible EMPC framework arises if we replace the terminal constraint $x(N) = x_s$ by $x(N) \in X_f$, where X_f is compact and contains x_s in its interior. At the same time we modify the cost function appending a *terminal cost* term.

Terminal region and cost

The new problem is

$$V^{\star}_{N,p}(x) = \min_{\pi} V_{N,p}(x, \pi),$$

subject to

$$x(k+1) = f(x(k), u(k)), \forall k \in \mathbb{N}_{[0,N-1]}$$
$$(x(k), u(k)) \in Z, \forall k \in \mathbb{N}_{[0,N-1]}$$
$$x(N) \in X_f, \text{ and } x(0) = x,$$

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&x(N) \in X_f, \text{ and } x(0) = x,
\end{aligned}$$

and $V_{N,p}(x,\pi)$ has the following form

$$V_{N,p}(x,\pi) = \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N)).$$

Terminal region and cost

The admissible set $Z_{N,p}$ is given by

$$\begin{aligned} Z_{N,p} &:= \left\{ (x,\pi) \;\middle|\; \begin{array}{l} \exists x(1),\ldots,x(N), x(k+1) = f(x(k),u(k)), \\ (x(k),u(k)) \in Z, \forall k \in \mathbb{N}_{[0,N-1]}, \\ x(N) \in X_f, x(0) = x \end{array} \right\} \\ &= \operatorname{dom} V^{\star}_{N,p}, \end{aligned}$$

and

$$X_{N,p} := \{x : \exists \pi : (x,\pi) \in Z_{N,p}\} = \operatorname{proj}_x Z_{N,p}.$$

Stabilizing conditions

There is a controller $\kappa_f : X_f \to \mathbb{R}^m$ so that

$$
\begin{aligned}
&V_f(f(x, \kappa_f(x))) - V_f(x) \leq -\ell(x, \kappa_f(x)) + \ell(x_s, u_s), \\
&(x, \kappa_f(x)) \in Z_{N,p},
\end{aligned}
$$

for all $x \in X_f$.

Question. Is it $\kappa_f(x) \in U_0^\star(x)$?

Stability conditions

Theorem [Angeli *et al.* '12] Under the above stabilizing conditions, if the system is *strictly dissipative* wrt $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$, then x_s is an asymptotically stable equilibrium point with domain of attraction $X_{N,p}$.

Proof.
For the proof we construct an equivalent optimisation problem defining

$$\tilde{V}_{N,p} := \sum_{k=0}^{N-1} L(x(k),u(k)) + \tilde{V}_f(x(N)),$$

with $\tilde{V}_f(x) := V_f(x) + \lambda(x) - \lambda(x_s)$. Details: Angeli *et al.* '12. □

* Construction of V_f, κ_f and X_f

How can we cosntruct V_f, κ_f and X_f so that

$$V_f(f(x,\kappa_f(x))) - V_f(x) \leq -\ell(x,\kappa_f(x)) + \ell(x_s,u_s),$$
$$(x,\kappa_f(x)) \in Z_{N,p},$$

for all $x \in X_f$?

The answer is not trivial and there exist various approaches; see Amrit *et al.* 2011.

* Construction of V_f, κ_f and X_f

Assumption. Functions f and ℓ are twice continuously differentiable, $f(0,0) = 0$ and $x_s = u_s = 0$. The linearised system

$$\bar{x}(k+1) = A\bar{x}(k) + Bu(k),$$

with $A = f_x(0,0)$ and $B = f_u(0,0)$ is stabilisable, so there exists a linear gain K so that $A_K = A + BK$ is stable.

* Construction of V_f, κ_f and X_f

Define $\bar{\ell}(x) := \ell(x, Kx) - \ell(0,0)$. Find $Q^\star$ so that

$$x'(Q^\star - \bar{\ell}_{xx}(x))x \geq 0, \forall x \in X,$$

Define $Q := Q^\star + \alpha I$, for some $\alpha > 0$. Define $q = \bar{\ell}_x(0)$. Choose P to be the solution of the Lyapunov equation

$$A_K' P A_K - P = -Q.$$

Define $p := q'(I - A_K)^{-1}$. Take a ball $\mathcal{B}_\delta \subseteq \{x \in \mathbb{R}^n : (x, Kx) \in Z\}$. Define $V_f(x) := \frac{1}{2}x'Px + p'x$. Take $\beta > 0$ so that $X_f := \mathrm{lev}_{\leq\beta} V_f \subseteq \mathcal{B}_\delta$, and $\kappa_f(x) := Kx$.

End of third section

Conclusions.

1. The rotated cost is equivalent to the original cost
2. Strict dissipativity entails as. stability
3. We may enforce stability by adding a PD function $\alpha(x, u)$ to the stage cost
4. We may replace $x(N) = x_s$ by $x(N) \in X_f$ and then
5. To have as. stability we need to add a terminal cost V_f and draw an assumption about V_f over X_f

IV. Averagely constrained MPC

Asymptotic averaging operator

Take $v \in \ell^\infty(\mathbb{R}^{n_v})$, that is $v = \{v(i)\}_{i\in\mathbb{N}}$ and there is a $M \geq 0$ so that for all $i \in \mathbb{N}$, $\|v(i)\| \leq M$. We define

$$\mathrm{Av}[v] = \left\{ \bar{v} \in \mathbb{R}^{n_v} \mid \exists \{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}, \text{s.t. } \frac{\sum_{k=0}^{t_n} \nu(k)}{t_n + 1} \xrightarrow{n} \bar{v} \right\}.$$

Asymptotic averaging operator

Examples.

Take $v = \{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$. Then, verify that $\mathrm{Av}[v] = 0$.

For $v = \{0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, \ldots\ldots\}$, we can verify that $\mathrm{Av}[v] = [1/3, 1/2]$.

Asymptotic averaging operator

Properties.

For $v \in \ell^\infty(\mathbb{R}^{n_v})$ define the P-shifted variant of v as $w = \{w(j)\}_{k\in\mathbb{N}}$ with $w(j) = v(j+P)$. Then, $\mathrm{Av}[v] = \mathrm{Av}[w]$ and

$$\mathrm{Av}\left(\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]\right) \subseteq \{\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \mathbb{R}^{2n_v} : v_1 = v_2\}.$$

Average constraints

Consider the system

$$x(k+1) = f(x(k), u(k)),$$
$$y(k) = h(x(k), u(k)).$$

with $h : Z \to \mathbb{R}^p$. The following constraints are imposed:

$$\mathrm{Av}[y] \in Y,$$

where $Y \subseteq \mathbb{R}^p$ is *closed*, *convex* and contains $h(x_s, u_s)$.

EMPC with average constraints

Our goal is to design a receding horizon control strategy with:

- ✓ Recursive feasibility
- ✓ [Performance bounds] $\mathrm{Av}[\ell(x,u)] \subseteq (-\infty, \ell(x_s, u_s)]$,
- ✓ [Constraints satisfaction] $(x(k), u(k)) \in Z$, for $k \in \mathbb{N}$,
- ✓ [Asymptotic average constraints] $\mathrm{Av}[y] \in Y$.

Averagely constrained EMPC

To this end, at time t we sove the problem:

$$V^{\star}_{N,a}(x;t) = \min_{\pi} \sum_{k=0}^{N-1} \ell(x(k), u(k)),$$

subject to the constraints:

$$x(k+1) = f(x(k), u(k)), \forall k \in \mathbb{N}_{[0,N-1]},$$
$$(x(k), u(k)) \in Z, \forall k \in \mathbb{N}_{[0,N-1]},$$
$$x(N) = x_s, x(0) = x,$$
$$\sum_{k=0}^{N-1} h(x(k), u(k)) \in Y_t.$$

where Y_t is time-varying.

Averagely constrained EMPC

. . . where[4]

$$Y_{t+1} = Y_t \oplus Y \ominus \{h(x(t), u(t))\},$$

and

$$Y_0 = NY + \bar{Y},$$

where $\bar{Y}$ is any compact convex set containing the origin.

We'll prove that: The resulting MPC controller is **recursively feasible** and all **requirements are satisfied** for the closed-loop system.

[4] $C \ominus \{z\} := \{y : y + z \in C\}$.

Averagely constrained EMPC

Define

$$Z_{N,a}(t) := \left\{ (x,\pi) \;\middle|\; \begin{array}{l} \exists x(1),\ldots,x(N), x(k+1) = f(x(k),u(k)), \\ (x(k),u(k)) \in Z, \forall k \in \mathbb{N}_{[0,N-1]}, \\ x(N) = x_s, x(0) = x \\ \sum_{k=0}^{N-1} h(x(k),u(k)) \in Y_t \end{array} \right\}$$
$$= \operatorname{dom} V_N^\star(\cdot,t),$$

and

$$X_{N,a}(t) := \{x : \exists \pi : (x,\pi) \in Z_{N,a}(t)\}$$
$$= \operatorname{proj}_x Z_{N,a}(t).$$

The receding horizon controller is a mapping $\kappa_{N,a} : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$ such that $(x, \kappa_{N,a}(x,t)) \in Z_{N,a}(t)$ whenever $x \in X_{N,a}(t)$.

Recursive feasibility of AC-EMPC

The averagely constrained EMPC is recursively feasible: if $x(t) \in X_{N,a}(t)$, then $x(t+1) \in X_{N,a}(t+1)$.

Proof. *Exercise.* show that $x(t+1) = f(x, \kappa_{N,a}(x,t)) \in X_{N,a}(t+1)$ given that take $x(t) = x \in X_{N,a}(t)$. □

Asymptotic average performance

The asymptotic average performance of AC-EMPC is no worse than the cost of best admissible steady state, that is

$$\mathrm{Av}[\ell(x,u)] \subseteq (-\infty, \ell(x_s,u_s)]$$

Proof. *Exercise.* we need to prove that

$$\limsup_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} \le \ell(x_s,u_s).$$

Start by verifying that $V_{N,a}^{\star}(x^+) - V_{N,a}^{\star}(x) \le \ell(x_s,u_s) - \ell(x,u)$. □

Average constraints satisfaction

$\mathrm{Av}[y] \subseteq Y$.

Proof. The EMPC controller produces the following sequence of sets Y_t:

$$Y_t = \bar{Y} \oplus (t+N)Y \ominus \{\sum_{k=0}^{t-1} y(k)\}.$$

According to the problem constraints:

$$\sum_{k=0}^{N-1} h(x(k), u(k)) \in Y_t$$

$$\Leftrightarrow \sum_{k=0}^{N-1} h(x(k), u(k)) + \sum_{k=0}^{t-1} y(k) \in \bar{Y} \oplus (t+N)Y$$

Average constraints satisfaction

Proof (cont'd).
Thus

$$\sum_{k=0}^{N-1} h(x(k), u(k)) \in \bigoplus_{k=0}^{N-1} h(Z),$$

which is a compact set since Z is compact and h is continuous. Take $\{t_n\}_n \subseteq \mathbb{N}$ so that the limit $\lim_n \sum_{k=0}^{t_n} y(k)/t_n$ exists; then

$$\lim_n \sum_{k=0}^{t_n} \frac{y(k)}{t_n} \in \lim_n \frac{\bar{Y} + (t_n + 1 + N)Y}{t_n + 1} = Y,$$

thus $\mathrm{Av}[y] \subseteq Y$. □

End of fourth section

Conclusions.

1. We introduced the asymptotic average operator $\mathrm{Av}[\cdot]$
2. This allows us to impose constraints on the asymptotic average of the output y
3. and the asymptotic average of the cost as well
4. We formulated an asymptotically constrained EMPC and
5. we showed that it is recursively feasible and satisfies the prescribed as. average constraints

V. Average performance

Optimally operated at steady state

A control system $x^+ = f(x, u)$ is said to be **optimally operated at steady state** with respect to a stage cost $\ell : Z \to \mathbb{R}$ if[5]

$$\mathrm{Av}[\ell(x, u)] \subseteq [\ell(x_s, u_s), +\infty),$$

for any feasible solution $(x(k), u(k)) \in Z$ for $k \in \mathbb{N}$.

[5] In this section, without loss of generality, we assume that there are no average constraints imposed on the system – that is, constraints of the form $\mathrm{Av}[y] \subseteq Y$ – i.e., let us assume $h \equiv 0$ and $Y = \{0\}$.

Optimally operated at steady state

A control system $x^+ = f(x, u)$ is said to be **optimally operated at steady state** with respect to a stage cost $\ell : Z \to \mathbb{R}$ if[5]

$$\mathrm{Av}[\ell(x, u)] \subseteq [\ell(x_s, u_s), +\infty),$$

for any feasible solution $(x(k), u(k)) \in Z$ for $k \in \mathbb{N}$. Equivalently:

$$\liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k), u(k))}{T} \geq \ell(x_s, u_s).$$

[5] In this section, without loss of generality, we assume that there are no average constraints imposed on the system – that is, constraints of the form $\mathrm{Av}[y] \subseteq Y$ – i.e., let us assume $h \equiv 0$ and $Y = \{0\}$.

Suboptimally operated off steady state

A control system $x^+ = f(x, u)$ is said to be **suboptimally operated off steady state** with respect to a stage cost $\ell : Z \to \mathbb{R}$ if it is optimally operated at steady state and either of the following holds

1. $\mathrm{Av}[\ell(x, u)] \subseteq (\ell(x_s, u_s), +\infty)$,
2. $\liminf_{k\to\infty} \|x(k) - x_s\| = 0$.

Some definitions

Define

$$Z_0 := \left\{ (x,u) \in Z \;\middle|\; \begin{array}{l} \exists(\mathbf{z},\mathbf{v}), (z(0), v(0)) = (x,u) \\ z(k+1) = f(z(k), v(k)), \\ (z(k), v(k)) \in Z \end{array} \right\},$$

and

$$X_0 := \operatorname{proj}_x Z_0.$$

Available storage function

For $x \in X_0$, we define the **available storage** function with respect to a supply rate $s : Z \to \mathbb{R}$ as follows

$$S(x) := \sup_{\substack{T \geq 0 \\ z(0)=x \\ z(k+1)=f(z(k),v(k)),k\in\mathbb{N} \\ (z(k),v(k))\in Z,k\in\mathbb{N}}} \sum_{k=0}^{T-1} -s(z(k), v(k))$$

Remark: $S(x) \geq 0$, for all $x \in X$.

Dissipativity conditions

Theorem [Willems '72] . System $x^+ = f(x, u)$ subj. to the constraints $(x, u) \in Z$ is *dissipative* with respect to the supply rate $s : Z \to \mathbb{R}$ if and only if S is bounded on X_0. Furthermore, S is a storage function, i.e.,

$$S(f(x, u)) - S(x) \leq s(x, u), \forall (x, u) \in Z.$$

Conditions for optimal operation at steady state

The available storage S for the supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$ is bounded on X_0 iff there is a finite $c \in \mathbb{R}$ so that

$$\inf_{\substack{T \geq 0 \\ z(0)=x \\ z(k+1)=f(z(k),v(k)),k\in\mathbb{N} \\ (z(k),v(k))\in Z,k\in\mathbb{N}}} \sum_{k=0}^{T-1} \ell(x_s,u_s) - \ell(z(k),v(k)) \geq c,$$

for all $x \in X_0$. Recall that a system is optimally operated at steady state if (by definition)

$$\liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} \geq \ell(x_s,u_s).$$

Clearly, dissipativity $\Rightarrow$ optimal operation at steady state (but the converse is not true).

Conditions for suboptimal operation off steady state

Our system is suboptimally operated off steady state if it is strictly dissipative.

Proof.
Indeed, we can show that

$$\liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k)) - \rho(x(k))}{T} \geq \ell(x_s,u_s)$$

$$\Rightarrow \liminf_{T\to\infty} \frac{\rho(x(k))}{T} \leq \liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k)) - \rho(x(k))}{T} - \ell(x_s,u_s),$$

from which it follows that the system if suboptimally operated off steady state. □

Averagely constrained systems

What if the system trajectories are averagely constrained?

$$\mathrm{Av}[h(x,u)] \subseteq Y := \{y : H_y y \leq K_y\}.$$

The same theory applies, but using the following supply rate:

$$s(x,u) := \ell(x,u) - \ell(x_s,u_s) + \bar{\lambda}'(H_y h(x,u) - K_y),$$

for some vector $\bar{\lambda} \geq 0$.

Averagely constrained systems

And the available storage function becomes

$$S(x) := \sup_{\substack{T\geq 0,\ z(0)=x \\ z(k+1)=f(z(k),v(k)),k\in\mathbb{N} \\ (z(k),v(k))\in Z,k\in\mathbb{N} \\ \mathrm{Av}[h(z,v)]\subseteq Y}} \sum_{k=0}^{T-1} -s(z(k),v(k)).$$

Averagely constrained systems

Theorem [Angeli *et al.* '12 & Müller *et al.* '12] . A system $x^+ = f(x,u)$ with constraints $(x,u) \in Z$ and average constraints $\mathrm{Av}[h(x,u)] \subseteq Y$ is **optimally operated at steady state** (suboptimally operated off steady state) if it is **dissipative**[6] (strictly dissipative) **on averagely constrained trajectories**.

[6] Using the supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s) + \bar{\lambda}'(H_y h(x,u) - K_y)$.

End of fifth section

Conclusions.

1. We introduced the notions of *optimal operation at steady state* and *suboptimal operation off steady state*
2. We elaborated on the powerful notion of available storage and
3. presented Willem's dissipativity theorem
4. We used the available storage function to determine conditions for OOSS and SOOSS

VI. Conclusions

Conclusions

1. EMPC combines process economics and control in a natural way
2. Stability not for granted
3. Dissipativity plays a crucial role in proving stability and optimal operation at steady state
4. Standard MPC practices can still be used

Extensions

1. EMPC without a terminal constraint (Grüne, 2013)
2. Generalized terminal state constraint: unifying tracking and economic MPC (Fagiano and Teel, 2013)
3. A Lyapunov function for EMPC (Diehl *et al.*, 2011)
4. Scenario-based EMPC (Bø and Johansen, 2014)

Topics for research

1. Robust EMPC: how resilient is EMPC to disturbances and how may disturbances affect average performance?
2. EMPC for uncertain systems in presence of probabilistic information (e.g., Markovian switching systems)
3. Lyapunov theorems for averagely constrained EMPC
4. Applications of EMPC

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